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Hung Nguyen-Schäfer
Jan-Philip Schmidt

Tensor Analysis and Elementary Differential Geometry for Physicists and Engineers

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In memory of
Gregorio Ricci-Curbastro (1853–1925) and
Tullio Levi-Civita (1873–1941),
who invented Tensor Calculus, for which
Elwin Bruno Christoffel (1829–1900)
had prepared the ground;

Carl Friedrich Gauss (1777–1855) and
Bernhard Riemann (1826–1866),
who invented Differential Geometry

Preface

This book represents a joint effort by a research engineer and a mathematician. The initial idea for it arose from our many years of experience in the automotive industry, advanced research development, and of course from our common research interest in applied mathematics, physics, and engineering. The main reason for this cooperation is the fact that mathematicians generally approach problems using mathematical rigor, but which need not always be practically applicable; at the same time, engineers usually deal with problems involving applied mathematics, which must ultimately work in the "real world" of industry. Having recognized that what mathematicians consider rigor can be more like rigor mortis for engineers and physicists, this joint effort proposes a compromise between the mathematical rigors and less rigorous applied mathematics, incorporating different points of view.

Our main aim is to bridge the mathematical gap between where physics and engineering mathematics end and where tensor analysis begins, which we do with the help of a powerful and user-friendly tool often employed in computational methods for physical and engineering problems in any general curvilinear coordinate system. However, tensor analysis has certain strict rules and conventions that must unconditionally be adhered to. This book is intended to support research scientists and practicing engineers in various fields who use tensor analysis and differential geometry in the context of applied physics, electrical and mechanical engineering. Moreover, it can also be used as a textbook for graduate students in applied physics and engineering.

Tensor analysis and differential geometry were pioneered by great mathematicians in the late nineteenth century, chiefly Curbastro, Levi-Civita, Christoffel, Ricci, Gauss, Riemann, Weyl, and Minkowski, and later promoted by well-known theoretical physicists in the early twentieth century, mainly Einstein, Dirac, Heisenberg, and Fermi, working on relativity and quantum mechanics. Since then, tensor analysis and differential geometry have taken on an increasingly important role in the mathematical language used in the modern physics of quantum mechanics and general relativity, and in many applied sciences fields. They have also been applied to computational mechanical and electrical engineering in classical mechanics, aero and vibroacoustics, computational fluid dynamics (CFD), continuum mechanics, electrodynamics, and cybernetics.

Approaching the topics of tensors and differential geometry in a mathematically rigorous way would not only require an immense amount of effort, it would not be practical for working engineers and applied physicists. As such, we decided to present these topics in a comprehensive and approachable way that will show readers how to work with tensors and differential geometry and to apply them to modeling the physical and engineering world. This book also includes numerous examples with solutions and concrete calculations in order to guide readers through these complex topics step-by-step. For the sake of simplicity and keeping the target audience in mind, we deliberately neglect certain aspects of mathematical rigor in this book, discussing them informally instead. Therefore, those readers who are more mathematically interested should consult the recommended literature.

We would like to thank Mmes Hestermann-Beyerle and Kollmar-Thoni at Springer Heidelberg for their helpful suggestions and valued cooperation during the preparation of this book.

Hung Nguyen-Schäfer
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Dr. Jan-Philip Schmidt is a mathematician. He studied mathematics, physics, and economics at the University of Heidelberg, Germany. He received a Ph.D. degree in Mathematics from the University of Heidelberg in 2012. His doctoral thesis was funded by a research fellowship from the Heidelberg Academy of Sciences, in collaboration with the Interdisciplinary Center for Scientific Computing (IWR) at the University of Heidelberg. His academic working experience comprises several research visits in France and Israel, as well as project works at the Max-Planck-Institute for Mathematics in the Sciences (MPIMIS) in Leipzig, and at the Max-Planck-Institute for Molecular Genetics (MPIMG) in Berlin. He also worked as a research associate in the AVACS program at Saarland University, Cluster of Excellence (MMCI).

Chapter 1
General Basis and Bra–Ket Notation

We begin this chapter by reviewing some mathematical backgrounds dealing with coordinate transformations and general basis vectors in general curvilinear coordinates. Some of these aspects will be informally discussed for the sake of simplicity. Therefore, those readers interested in more in-depth coverage should consult the literature recommended under Further Reading. To simplify notation, we will denote a basis vector simply as **basis** in the following section.

We assume that the reader has already had fundamental backgrounds about vector analysis in finite N-dimensional spaces with the general bases of curvilinear coordinates. However, this topic is briefly recapitulated in Appendix E.

1.1 Introduction to General Basis and Tensor Types

A physical state generally depending on N different variables is defined as a point $\mathrm{P}(u^1,\ldots, u^N)$ that has N-independent coordinates of u^i. At changing the variables, such as time, locations, and physical characteristics, the physical state P moves from one position to other positions. All relating positions generate a set of points in an N-dimension space. This is the point space with N dimensions (N-point space). Additionally, the state change between two points could be described by a vector **r** connecting them that obviously consists of N-vector components. All state changes are displayed by the vector field that belongs to the vector space with N dimensions (N-vector space). Generally, a differentiable hypersurface in an N-dimensional space with general curvilinear coordinates $\{u^i\}$ for $i = 1, 2,\ldots, N$ is defined as a differentiable $(N - 1)$-dimensional subspace with a codimension of one. Subspaces with any codimension are called manifolds of an N-dimensional space (cf. Appendix E).

Physically, the vector length does not change in any coordinate system. However, its components depend on the coordinate system. That means the vector components vary as the coordinate system changes. Generally, tensors are a very useful tool applied to the coordinate transformations between two general curvilinear coordinate systems in finite N-dimensional real spaces. The exemplary second-order tensor can be defined as a multilinear functional **T** that maps an

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Table 1.1 Different types of tensors

Tensors of	0-order T	1-order T_i	2-order T_{ij}	3-order T_{ijk}	Higher-order $T_{ij\ldots pq}$
Scalar a $\in \mathbf{R}$	X				
Vector $\mathbf{v} \in \mathbf{R}^N$		X			
Matrix $\mathbf{M} \in \mathbf{R}^N \times \mathbf{R}^N$			X		
Bra $\langle \mathbf{B}\|$ and Ket $\|\mathbf{A}\rangle \in \mathbf{R}^N, \mathbf{R}^N \times \mathbf{R}^N$		X	X		
Levi-Civita symbols $\varepsilon_{ijk} \in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N$				X	(X)
Higher-order tensors $T_{ij\ldots pq} \in \mathbf{R}^N \times \cdots \times \mathbf{R}^N$					X

arbitrary vector in a vector space into the image vector in another vector space. Like vectors, tensors do not change in any coordinate system and the tensor components only depend on the relating transformed coordinate systems. Therefore, the tensor components change as the coordinate system varies.

Scalars, vectors, and matrices are special types of tensors:

- scalar (invariant) is a zero-order tensor,
- vector is a first-order tensor,
- matrix is arranged by a second-order tensor,
- bra and ket are first- and second-order tensors,
- Levi-Civita permutation symbols in a three-dimensional space are third-order pseudo-tensors (Table 1.1).

We consider two important spaces in tensor analysis: first, Euclidean N-spaces with orthogonal and curvilinear coordinate systems; second, general curvilinear Riemannian manifolds of dimension N (cf. Appendix E).

1.2 General Basis in Curvilinear Coordinates

We consider three covariant basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ to the general curvilinear coordinates (u^1, u^2, and u^3) at the point P in Euclidean space $\mathbf{E}^3$. The non-orthonormal basis $\mathbf{g}_i$ can be calculated from the orthonormal bases ($\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$) in Cartesian coordinates $x^j = x^j(u^i)$ using Einstein summation convention (cf. Sect. 2.1).

$$\begin{aligned} \mathbf{g}_i &\equiv \frac{\partial \mathbf{r}}{\partial u^i} = \sum_{j=1}^{3} \frac{\partial \mathbf{r}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} \equiv \frac{\partial \mathbf{r}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} \\ &= \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \quad \text{for} \quad j = 1, 2, 3 \end{aligned} \tag{1.1}$$

The metric coefficients can be calculated by the scalar products of the covariant and contravariant bases in general curvilinear coordinates with non-orthonormal bases (i.e., non-orthogonal and non-unitary). There are the covariant, contravariant, and mixed metric coefficients g_{ij}, g^{ij}, and g_i^j, respectively.

$$\begin{aligned} g_{ij} &= g_{ji} = \mathbf{g}_i \cdot \mathbf{g}_j = \mathbf{g}_j \cdot \mathbf{g}_i \neq \delta_i^j \\ g^{ij} &= g^{ji} = \mathbf{g}^i \cdot \mathbf{g}^j = \mathbf{g}^j \cdot \mathbf{g}^i \neq \delta_i^j \\ g_i^j &= \mathbf{g}_i \cdot \mathbf{g}^j = \mathbf{g}_j \cdot \mathbf{g}^i = \delta_i^j \end{aligned} \tag{1.2}$$

where the Kronecker delta, δ_i^j, is defined as

$$\delta_i^j \equiv \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases}$$

Similarly, the bases of the orthonormal coordinates can be written in the non-orthonormal bases of the curvilinear coordinates $u^i = u^i(x^j)$.

$$\begin{aligned} \mathbf{e}_j &\equiv \frac{\partial \mathbf{r}}{\partial x^j} = \sum_{i=1}^{3} \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial u^i}{\partial x^j} \equiv \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial u^i}{\partial x^j} \\ &= \mathbf{g}_i \frac{\partial u^i}{\partial x^j} \quad \text{for} \quad i = 1, 2, 3 \end{aligned} \tag{1.3}$$

The covariant and contravariant bases of the orthonormal coordinates (orthogonal and unitary bases) have the following properties:

$$\begin{aligned} \mathbf{e}_i \cdot \mathbf{e}_j &= \mathbf{e}_j \cdot \mathbf{e}_i = \delta_i^j; \\ \mathbf{e}^i \cdot \mathbf{e}^j &= \mathbf{e}^j \cdot \mathbf{e}^i = \delta_i^j; \\ \mathbf{e}^i \cdot \mathbf{e}_j &= \mathbf{e}_j \cdot \mathbf{e}^i = \delta_i^j. \end{aligned} \tag{1.4}$$

The contravariant basis $\mathbf{g}^k$ of the curvilinear coordinate u^k is perpendicular to the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$ at the given point P, as shown in Fig. 1.1. The contravariant basis $\mathbf{g}^k$ can be defined as

$$\alpha \mathbf{g}^k \equiv \mathbf{g}_i \times \mathbf{g}_j \equiv \frac{\partial \mathbf{r}}{\partial u^i} \times \frac{\partial \mathbf{r}}{\partial u^j} \tag{1.5}$$

where

α is a scalar factor;

$\mathbf{g}^k$ is the contravariant basis of the curvilinear coordinate of u^k.

Multiplying Eq. (1.5) by the covariant basis $\mathbf{g}_k$, the scalar factor α results in

$$\begin{aligned} (\mathbf{g}_i \times \mathbf{g}_j) \cdot \mathbf{g}_k &= \alpha \mathbf{g}^k \cdot \mathbf{g}_k = \alpha \delta_k^k = \alpha \\ &\equiv [\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] \end{aligned} \tag{1.6}$$

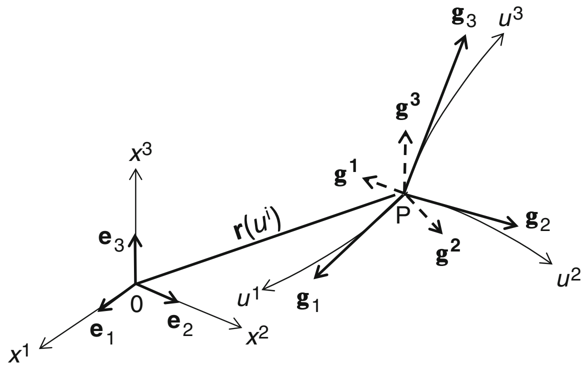

Fig. 1.1 Covariant and contravariant bases of curvilinear coordinates

The expression in the square brackets is called the scalar triple product.

Therefore, the contravariant bases of the curvilinear coordinates result from Eqs. (1.5 and 1.6).

$$\mathbf{g}^i = \frac{\mathbf{g}_j \times \mathbf{g}_k}{[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]}; \quad \mathbf{g}^j = \frac{\mathbf{g}_k \times \mathbf{g}_i}{[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]}; \quad \mathbf{g}^k = \frac{\mathbf{g}_i \times \mathbf{g}_j}{[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]} \tag{1.7}$$

Obviously, the relation of the covariant and contravariant bases results from Eq. (1.7).

$$\mathbf{g}^k \cdot \mathbf{g}_i = \frac{(\mathbf{g}_i \times \mathbf{g}_j) \cdot \mathbf{g}_i}{[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]} = \delta_i^k \tag{1.8}$$

where δ_i^k is the Kronecker delta.

The scalar triple product is an invariant under cyclic permutation; therefore, it has the following properties:

$$(\mathbf{g}_i \times \mathbf{g}_j) \cdot \mathbf{g}_k = (\mathbf{g}_k \times \mathbf{g}_i) \cdot \mathbf{g}_j = (\mathbf{g}_j \times \mathbf{g}_k) \cdot \mathbf{g}_i \tag{1.9}$$

Furthermore, the scalar triple product of the covariant bases of the curvilinear coordinates can be calculated (Nayak 2012).

$$[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = \varepsilon_{ijk} \frac{\partial x^i}{\partial u^1} \frac{\partial x^j}{\partial u^2} \frac{\partial x^k}{\partial u^3} = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} \equiv J \tag{1.10}$$

where J is the Jacobian, determinant of the covariant basis tensor; ε_{ijk} is the Levi-Civita symbols in Eq. (A.5), cf. Appendix A.

Squaring the scalar triple product in Eq. (1.10), one obtains

$$
\begin{aligned}
[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]^2 &= \begin{vmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} & \dfrac{\partial x^1}{\partial u^3} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} & \dfrac{\partial x^2}{\partial u^3} \\ \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} & \dfrac{\partial x^3}{\partial u^3} \end{vmatrix}^2 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \\
&= |g_{ij}| \equiv g = J^2
\end{aligned}
\tag{1.11}
$$

where $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ are the covariant metric coefficients.

Thus, the scalar triple product of the covariant bases results in

$$
\begin{aligned}
[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] &= (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 \\
&= \sqrt{g} = J
\end{aligned}
\tag{1.12}
$$

The covariant and contravariant bases of the orthogonal cylindrical and spherical coordinates will be studied in the following subsections.

1.2.1 Orthogonal Cylindrical Coordinates

Cylindrical coordinates (r, θ, and z) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 1.2 shows a point P in the cylindrical coordinates (r, θ, z), which is embedded in the orthonormal Cartesian coordinates (x^1, x^2, and x^3). However, the cylindrical coordinates change as the point P varies.

The vector **OP** can be written in Cartesian coordinates (x^1, x^2, x^3):

$$
\begin{aligned}
\mathbf{R} &= (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + z\,\mathbf{e}_3 \\
&\equiv x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3
\end{aligned}
\tag{1.13}
$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$	are the orthonormal bases of Cartesian coordinates;
θ	is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 are used for r, θ, and z, respectively. Therefore, the coordinates of $P(u^1, u^2, u^3)$ can be expressed in Cartesian coordinates:

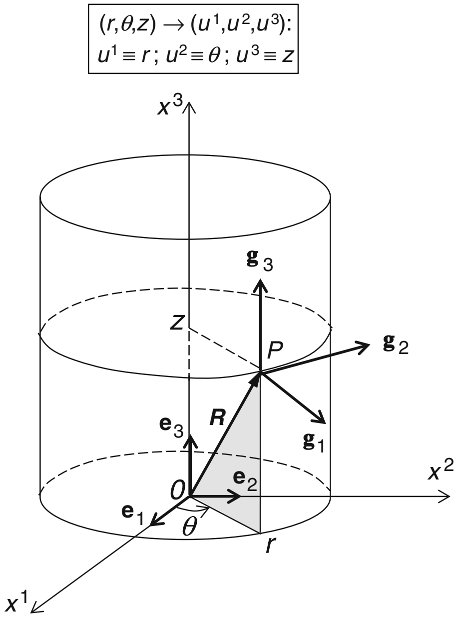

Fig. 1.2 Covariant bases of orthogonal cylindrical coordinates

$$P\left(u^1, u^2, u^3\right) = \begin{Bmatrix} x^1 = r\cos\theta \equiv u^1 \cos\ u^2 \\ x^2 = r\sin\theta \equiv u^1 \sin\ u^2 \\ x^3 = z \equiv u^3 \end{Bmatrix} \tag{1.14}$$

The covariant bases of the curvilinear coordinates can be computed from

$$\mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{R}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} = \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \quad \text{for} \quad j = 1, 2, 3 \tag{1.15}$$

The covariant basis matrix **G** can be calculated from Eq. (1.15).

$$\begin{aligned} \mathbf{G} &= [\,\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3\,] \\ &= \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{1.16}$$

The determinant of the covariant basis matrix **G** is called the Jacobian J.

$$|\mathbf{G}| \equiv J = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \tag{1.17}$$

The inversion of the matrix **G** yields the contravariant basis matrix $\mathbf{G}^{-1}$. The relation between the covariant and contravariant bases results from Eq. (1.8).

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j \text{ (Kronecker delta)} \tag{1.18a}$$

At det $(\mathbf{G}) \neq 0$ given from Eq. (1.17), Eq. (1.18a) is equivalent to

$$\mathbf{G}^{-1}\mathbf{G} = \mathbf{I} \tag{1.18b}$$

Thus, the contravariant basis matrix $\mathbf{G}^{-1}$ can be calculated from the inversion of the covariant basis matrix **G**, as given in Eq. (1.16).

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \begin{pmatrix} \frac{\partial u^1}{\partial x^1} & \frac{\partial u^1}{\partial x^2} & \frac{\partial u^1}{\partial x^3} \\ \frac{\partial u^2}{\partial x^1} & \frac{\partial u^2}{\partial x^2} & \frac{\partial u^2}{\partial x^3} \\ \frac{\partial u^3}{\partial x^1} & \frac{\partial u^3}{\partial x^2} & \frac{\partial u^3}{\partial x^3} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{pmatrix} \tag{1.19a}$$

The contravariant bases of the curvilinear coordinates can be written as

$$\mathbf{g}^i = \frac{\partial u^i}{\partial x^j}\mathbf{e}_j \quad \text{for} \quad j = 1, 2, 3 \tag{1.19b}$$

The calculation of the determinant and inversion matrix of **G** will be discussed in the following section.

According to Eq. (1.16), the covariant bases can be rewritten as

$$\begin{cases} \mathbf{g}_1 = (\cos\theta)\,\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2 + 0 \cdot \mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = 1 \\ \mathbf{g}_2 = (-r\sin\theta)\,\mathbf{e}_1 + (r\cos\theta)\,\mathbf{e}_2 + 0 \cdot \mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = r \\ \mathbf{g}_3 = 0 \cdot \mathbf{e}_1 + 0 \cdot \mathbf{e}_2 + 1 \cdot \mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = 1 \end{cases} \tag{1.20}$$

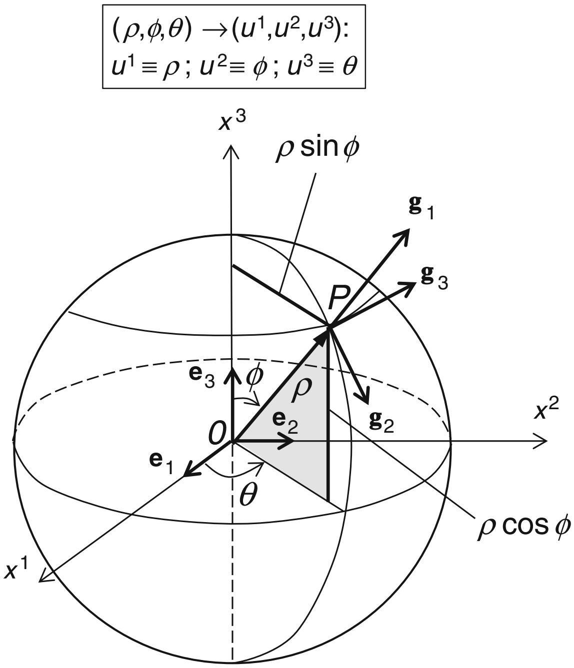

Fig. 1.3 Covariant bases of orthogonal spherical coordinates

The contravariant bases result from Eq. (1.19b).

$$\begin{cases} \mathbf{g}^1 = (\cos\theta)\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^1\right| = 1 \\ \mathbf{g}^2 = \left(-\dfrac{\sin\theta}{r}\right)\mathbf{e}_1 + \left(\dfrac{\cos\theta}{r}\right)\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^2\right| = \dfrac{1}{r} \\ \mathbf{g}^3 = 0\cdot\mathbf{e}_1 + 0\cdot\mathbf{e}_2 + 1\cdot\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^3\right| = 1 \end{cases} \tag{1.21}$$

Not only the covariant bases but also the contravariant bases of the cylindrical coordinates are orthogonal due to

$$\begin{aligned} &\mathbf{g}_i\cdot\mathbf{g}^j = \mathbf{g}^j\cdot\mathbf{g}_i = \delta_i^j \\ &\mathbf{g}_i\cdot\mathbf{g}_j = 0 \quad \text{for } i \neq j; \\ &\mathbf{g}^i\cdot\mathbf{g}^j = 0 \quad \text{for } i \neq j. \end{aligned}$$

1.2.2 Orthogonal Spherical Coordinates

Spherical coordinates (ρ, ϕ, and θ) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 1.3 shows a point P in the spherical coordinates (r, θ, and z) which is embedded in the orthonormal Cartesian coordinates (x^1, x^2, and x^3). However, the spherical coordinates change as the point P varies.

The vector **OP** can be rewritten in Cartesian coordinates (x^1, x^2, and x^3):

$$\begin{aligned}\mathbf{R} &= (\rho \sin\phi \cos\theta)\,\mathbf{e}_1 + (\rho \sin\phi \sin\theta)\mathbf{e}_2 + \rho\cos\phi\,\mathbf{e}_3 \\ &\equiv x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3\end{aligned} \tag{1.22}$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$	are the orthonormal bases of Cartesian coordinates;
ϕ	is the equatorial angle;
θ	is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 are used for ρ, ϕ, and θ, respectively. Therefore, the coordinates of $P(u^1, u^2, u^3)$ can be expressed in Cartesian coordinates:

$$P(u^1, u^2, u^3) = \begin{Bmatrix} x^1 = \rho\sin\phi\cos\theta \equiv u^1 \sin u^2 \cos u^3 \\ x^2 = \rho\sin\phi\sin\theta \equiv u^1 \sin u^2 \cos u^3 \\ x^3 = \rho\cos\phi \equiv u^1 \cos u^2 \end{Bmatrix} \tag{1.23}$$

The covariant bases of the curvilinear coordinates can be computed by means of

$$\begin{aligned}\mathbf{g}_i &= \frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{R}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} \\ &= \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \quad \text{for} \quad j = 1, 2, 3\end{aligned} \tag{1.24}$$

Thus, the covariant basis matrix **G** can be calculated from Eq. (1.24).

$$\begin{aligned}\mathbf{G} = [\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3] &= \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} \\ &= \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}\end{aligned} \tag{1.25}$$

$$|\mathbf{G}| \equiv J = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi \tag{1.26}$$

The determinant of the covariant basis matrix **G** is called the Jacobian J.

Similarly, the contravariant basis matrix $\mathbf{G}^{-1}$ is the inversion of the covariant basis matrix.

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \begin{pmatrix} \frac{\partial u^1}{\partial x^1} & \frac{\partial u^1}{\partial x^2} & \frac{\partial u^1}{\partial x^3} \\ \frac{\partial u^2}{\partial x^1} & \frac{\partial u^2}{\partial x^2} & \frac{\partial u^2}{\partial x^3} \\ \frac{\partial u^3}{\partial x^1} & \frac{\partial u^3}{\partial x^2} & \frac{\partial u^3}{\partial x^3} \end{pmatrix} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\phi\cos\theta & \rho\sin\phi\sin\theta & \rho\cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\left(\frac{\sin\theta}{\sin\phi}\right) & \left(\frac{\cos\theta}{\sin\phi}\right) & 0 \end{pmatrix} \tag{1.27a}$$

The contravariant bases of the curvilinear coordinates can be written as

$$\mathbf{g}^i = \frac{\partial u^i}{\partial x^j}\mathbf{e}_j \quad \text{for} \quad j = 1, 2, 3 \tag{1.27b}$$

The matrix product $\mathbf{G}^{-1} \cdot \mathbf{G}$ must be an identity matrix according to Eq. (1.18b).

$$\mathbf{G}^{-1}\mathbf{G} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\phi\cos\theta & \rho\sin\phi\sin\theta & \rho\cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\left(\frac{\sin\theta}{\sin\phi}\right) & \left(\frac{\cos\theta}{\sin\phi}\right) & 0 \end{pmatrix} \cdot \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{I} \tag{1.28}$$

According to Eq. (1.25), the covariant bases can be written as

$$\begin{aligned} \mathbf{g}_1 &= (\sin\phi\cos\theta)\,\mathbf{e}_1 + (\sin\phi\sin\theta)\,\mathbf{e}_2 + \cos\phi\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = 1 \\ \mathbf{g}_2 &= (\rho\cos\phi\cos\theta)\,\mathbf{e}_1 + (\rho\cos\phi\sin\theta)\mathbf{e}_2 - (\rho\sin\phi)\mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = \rho \\ \mathbf{g}_3 &= (-\rho\sin\phi\sin\theta)\mathbf{e}_1 + (\rho\sin\phi\cos\theta)\,\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = \rho\sin\phi \end{aligned} \tag{1.29}$$

The contravariant bases result from Eq. (1.27b).

$$\begin{aligned}
\mathbf{g}^1 &= (\sin\phi\cos\theta)\,\mathbf{e}_1 + (\sin\phi\sin\theta)\,\mathbf{e}_2 + \cos\phi\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^1\right| = 1 \\
\mathbf{g}^2 &= \left(\frac{1}{\rho}\cos\phi\cos\theta\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\cos\phi\sin\theta\right)\mathbf{e}_2 - \left(\frac{1}{\rho}\sin\phi\right)\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^2\right| = \frac{1}{\rho} \\
\mathbf{g}^3 &= \left(-\frac{1}{\rho}\frac{\sin\theta}{\sin\phi}\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\frac{\cos\theta}{\sin\phi}\right)\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^3\right| = \frac{1}{\rho\sin\phi}
\end{aligned} \tag{1.30}$$

Not only the covariant bases but also the contravariant bases of the spherical coordinates are orthogonal due to

$$\begin{aligned}
&\mathbf{g}_i\cdot\mathbf{g}^j = \mathbf{g}^j\cdot\mathbf{g}_i = \delta_i^j \\
&\mathbf{g}_i\cdot\mathbf{g}_j = 0 \quad \text{for} \quad i \neq j; \\
&\mathbf{g}^i\cdot\mathbf{g}^j = 0 \quad \text{for} \quad i \neq j.
\end{aligned}$$

1.3 Eigenvalue Problem of a Linear Coupled Oscillator

In the following subsection, we will give an example of the application of vector and matrix analysis to the eigenvalue problems in mechanical vibration. Figure 1.4 shows the free vibrations without damping of a three-mass system with the masses m_1, m_2, and m_3 connected by the springs with the constant stiffness k_1, k_2, and k_3. In the case of the small vibration amplitudes and constant spring stiffnesses, the vibrations can be considered linear. Otherwise, the vibrations are nonlinear for that the bifurcation theory must be used to compute the responses (Nguyen-Schäfer 2012).

Using Newton's second law, the homogenous vibration equations (free vibration equations) of the three-mass system can be written as (Nguyen-Schäfer 2012; Kraemer 1993; Muszýnska 2005; Vance 1988; Yamamoto and Ishida 2001):

$$\begin{aligned}
&m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0 \\
&m_2\ddot{x}_2 + k_2(x_2 - x_1) + k_3(x_2 - x_3) = 0 \\
&m_3\ddot{x}_3 + k_3(x_3 - x_2) = 0
\end{aligned} \tag{1.31}$$

Thus,

$$\begin{aligned}
&m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \\
&m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0 \\
&m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0
\end{aligned}$$

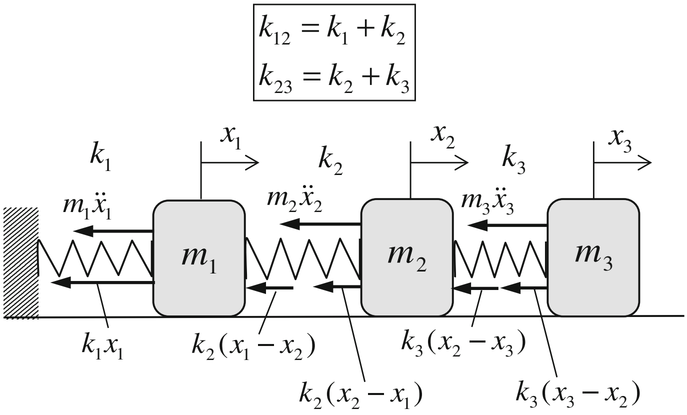

Fig. 1.4 Free vibrations of a three-mass system

Using the abbreviations of $k_{12} \equiv k_1 + k_2$ and $k_{23} \equiv k_2 + k_3$, one obtains

$$\begin{aligned} &m_1\ddot{x}_1 + k_{12}x_1 - k_2x_2 = 0 \\ &m_2\ddot{x}_2 - k_2x_1 + k_{23}x_2 - k_3x_3 = 0 \\ &m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0 \end{aligned}$$

The vibration equations can be rewritten in the matrix formulation:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \cdot \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{bmatrix} k_{12} & -k_2 & 0 \\ -k_2 & k_{23} & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.32}$$

Thus,

$$\begin{aligned} &\ddot{\mathbf{x}} + (\mathbf{M}^{-1}\mathbf{K})\mathbf{x} = \mathbf{0} \\ &\Leftrightarrow \ddot{\mathbf{x}} + \mathbf{A}\mathbf{x} = \mathbf{0} \end{aligned} \tag{1.33}$$

where

$$\mathbf{A} \equiv \mathbf{M}^{-1}\mathbf{K} = \begin{bmatrix} \frac{k_{12}}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_{23}}{m_2} & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3}{m_3} \end{bmatrix}$$

The free vibration response of Eq. (1.33) can be assumed as

$$\begin{aligned} &\mathbf{x} = \mathbf{X}e^{\lambda t} \\ &\Rightarrow \dot{\mathbf{x}} = \lambda(\mathbf{X}e^{\lambda t}) = \lambda\mathbf{x} \\ &\Rightarrow \ddot{\mathbf{x}} = \lambda^2(\mathbf{X}e^{\lambda t}) = \lambda^2\mathbf{x} \end{aligned} \tag{1.34}$$

where λ is the complex eigenvalue that is defined by

$$\lambda = \alpha \pm j\omega \in \mathbf{C} \tag{1.35}$$

in which ω is the eigenfrequency; α is the growth/decay rate (Nguyen-Schäfer 2012).

Substituting Eq. (1.34) into Eq. (1.33), one obtains the eigenvalue problem

$$(\mathbf{A} + \lambda^2\mathbf{I})\mathbf{X}\, e^{\lambda t} = \mathbf{0} \tag{1.36}$$

where $\mathbf{X}$ is the eigenvector relating to its eigenvalue λ; $\mathbf{I}$ is the identity matrix.

For any non-trivial solution of $\mathbf{x}$, the determinant of $(\mathbf{A} + \lambda^2\mathbf{I})$ must vanish.

$$\det\,(\mathbf{A} + \lambda^2\mathbf{I}) = \mathbf{0} \tag{1.37}$$

Equation (1.37) is called the characteristic equation of the eigenvalue. Obviously, this characteristic equation is a polynomial of λ^{2N} where N is the degrees of freedom (DOF) of the vibration system. In this case, the DOF equals 3 for a three-mass system in the translational vibration.

Solving the characteristic Eq. (1.37), one obtains 6 eigenvalues (= 2*DOF) for the vibration equations of the three-mass system. In the case without damping, the eigenvalues result in

$$\lambda_1 = \pm j\omega_1; \quad \lambda_2 = \pm j\omega_2; \quad \lambda_3 = \pm j\omega_3 \tag{1.38}$$

Substituting the eigenvalue λ_i into Eq. (1.36), one obtains the corresponding eigenvector $\mathbf{X}_i$.

$$(\mathbf{A} + \lambda_i^2\mathbf{I})\mathbf{X}_i = \mathbf{0} \quad \text{for} \quad i = 1, 2, 3 \tag{1.39}$$

The eigenvectors in Eq. (1.39) relating to the eigenvalues show the vibration modes of the system.

It is well known that N ordinary differential equations (ODEs) of second order can be transformed into $2N$ ODEs of first order using the simple trick of adding N identical ODEs of first order to the original ODEs.

$$\begin{pmatrix} \dot{\mathbf{x}} = \dot{\mathbf{x}} \\ \mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \dot{\mathbf{x}} = \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{x} \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix} \tag{1.40}$$

Substituting a new $(2N \times 1)$ vector of

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix} \Rightarrow \dot{\mathbf{z}} = \begin{pmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix} \tag{1.41}$$

into Eq. (1.40), the vibration equations of first order can be rewritten down

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & 0 \end{bmatrix} \cdot \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix} \Leftrightarrow \dot{\mathbf{z}} = \mathbf{B}\mathbf{z} \tag{1.42}$$

The free vibration response of Eq. (1.42) can be assumed as

$$\mathbf{z} = \mathbf{Z}e^{\lambda t} \Rightarrow \dot{\mathbf{z}} = \lambda(\mathbf{Z}e^{\lambda t}) = \lambda\mathbf{z} \tag{1.43}$$

where λ is the complex eigenvalue given in

$$\lambda = \alpha \pm j\omega \in \mathbf{C} \tag{1.44}$$

within ω is the eigenfrequency; α is the growth/decay rate.

Substituting Eq. (1.43) into Eq. (1.42), one obtains the eigenvalue problem

$$(\mathbf{B} - \lambda\mathbf{I})\mathbf{Z}e^{\lambda t} = \mathbf{0} \tag{1.45}$$

where $\mathbf{Z}$ is the eigenvector relating to its eigenvalue λ; $\mathbf{I}$ is the identity matrix.

For any non-trivial solution of $\mathbf{z}$, the determinant of $(\mathbf{B} - \lambda\mathbf{I})$ must vanish:

$$\det(\mathbf{B} - \lambda\mathbf{I}) = \mathbf{0} \tag{1.46}$$

Equation (1.46) is called the characteristic equation of the eigenvalue that is identical to Eq. (1.37). Obviously, this characteristic equation is a polynomial of λ^{2N}, where N is the degrees of freedom (DOF) of the vibration system. In this case, the DOF equals 3 because of the three-mass system.

Solving the characteristic Eq. (1.46), one obtains 6 eigenvalues (= 2*DOF) for the vibration equations of the three-mass system. In the case without damping, the eigenvalues result in

$$\lambda_1 = \pm j\omega_1; \quad \lambda_2 = \pm j\omega_2; \quad \lambda_3 = \pm j\omega_3 \tag{1.47}$$

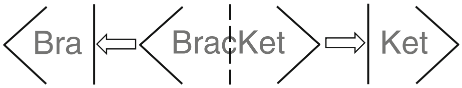

Fig. 1.5 Notation of bra and ket

Substituting the eigenvalue λ_i into Eq. (1.45), it gives the corresponding eigenvector $\mathbf{Z}_i$.

$$(\mathbf{B} - \lambda_i \mathbf{I})\mathbf{Z}_i = \mathbf{0} \quad \text{for} \quad i = 1, 2, 3 \tag{1.48}$$

The eigenvectors in Eq. (1.48) relating to the eigenvalues describe the vibration modes of the system.

1.4 Notation of Bra and Ket

The notation of bra and ket was defined by Dirac for applications in quantum mechanics and statistical thermodynamics (Dirac 1958). Bra and ket are tuples of independent coordinates in a finite N-dimensional space in Riemannian manifold (cf. Appendix E). The name of bra and ket comes from the angle bracket $\langle\ \rangle$, as shown in Fig. 1.5. Dividing the bracket into two parts, one obtains the left one called bra and the right one named ket.

In general, bra and ket can be considered as vectors, matrices, and high-order tensors. In contrast to vectors (first-order tensors), bra and ket have generally neither direction nor vector length in the point space. They are only a tuple of N coordinates (dimensions), such as of time, position, momentum, velocity, etc. Bra and ket are independent of any coordinate system, but their components depend on the relating basis of the coordinate system; that is, they are changed at the new basis by coordinate transformations. Therefore, the bra and ket notation is a powerful tool mostly used in quantum mechanics and statistical thermodynamics in order to describe a physical state as a point of N dimensions in a finite N-dimensional complex space.

Some examples of bra and ket can be written in different types:

$$\text{Ket vector } |\mathbf{K}\rangle = \begin{bmatrix} 1+i \\ -1 \\ 2-i \\ 1 \end{bmatrix} \rightarrow \text{Bra vector } \langle\mathbf{K}| = [\,(1-i) \quad -1 \quad (2+i) \quad 1\,];$$

$$\text{Ket matrix } |\mathbf{M}\rangle = \begin{bmatrix} 1+i & -2 \\ 1 & 2-i \end{bmatrix} \rightarrow \text{Bra matrix } \langle\mathbf{M}| = \begin{bmatrix} 1-i & 1 \\ -2 & 2+i \end{bmatrix}.$$

1.5 Properties of Kets

We denote the finite N-dimensional complex vector space by $\mathbf{C}^N$. A ket $|\mathbf{K}\rangle$ can be defined as an N-tuple of the coordinates $u^1,\ldots, u^N$: $\mathbf{K}(u^1,\ldots, u^N) \in \mathbf{C}^\mathbf{N}$. Given three arbitrary kets $|\mathbf{A}\rangle$, $|\mathbf{B}\rangle$, and $|\mathbf{C}\rangle \in \mathrm{C}^N$ and two scalars α, $\beta \in \mathbf{C}$, the following properties of kets result in Shankar (1980):

- Commutative property of ket addition:

$$|\mathbf{A}\rangle + |\mathbf{B}\rangle = |\mathbf{B}\rangle + |\mathbf{A}\rangle$$

- Distributive property of ket multiplication by a scalar addition:

$$(\alpha + \beta)|\mathbf{A}\rangle = \alpha|\mathbf{A}\rangle + \beta|\mathbf{A}\rangle$$

- Distributive property of multiplication of ket addition by a scalar:

$$\alpha(|\mathbf{A}\rangle + |\mathbf{B}\rangle) = \alpha|\mathbf{A}\rangle + \alpha|\mathbf{B}\rangle$$

- Associative property of ket addition:

$$|\mathbf{A}\rangle + (|\mathbf{B}\rangle + |\mathbf{C}\rangle) = (|\mathbf{A}\rangle + |\mathbf{B}\rangle) + |\mathbf{C}\rangle$$

- Associative property of ket multiplication by scalars:

$$\alpha(\beta|\mathbf{A}\rangle) = \beta(\alpha|\mathbf{A}\rangle) = \alpha\beta|\mathbf{A}\rangle$$

- Property of ket addition to the null ket $|\mathbf{0}\rangle$:

$$|\mathbf{A}\rangle + |\mathbf{0}\rangle = |\mathbf{A}\rangle$$

- Property of ket multiplication by the null scalar:

$$0 \cdot |\mathbf{A}\rangle = |\mathbf{0}\rangle$$

- Property of ket addition to an inverse ket $|-\mathbf{A}\rangle$:

$$|\mathbf{A}\rangle + |-\mathbf{A}\rangle = |\mathbf{A}\rangle - |\mathbf{A}\rangle = |\mathbf{0}\rangle.$$

1.6 Analysis of Bra and Ket

1.6.1 Bra and Ket Bases

Ket vector $|\mathbf{A}\rangle$ of the coordinates $(u^1,\ldots, u^N)$: $\mathbf{A}(a_1,\ldots, a_N) \in \mathbf{V}^N$ is the sum of its components in the orthonormal ket bases and can be written as

$$|\mathbf{A}\rangle \equiv \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ a_N \end{pmatrix} = \sum_{i=1}^{N} a_i|\mathbf{i}\rangle; \quad a_i \equiv (\alpha_i + \mathrm{j}\beta_i) \in \mathbf{C} \tag{1.49}$$

where
a_i is the ket component in the basis $\mathbf{i}$;
a_i is a complex number, $a_i \in \mathbf{C}$;
$|\mathbf{i}\rangle$ is the orthonormal basis of the coordinates $(u^1,\ldots, u^N)$

The ket bases of $\{|\mathbf{1}\rangle, |\mathbf{2}\rangle, \ldots, |\mathbf{N}\rangle\}$ in the coordinates $(u^1,\ldots, u^N)$ are column vectors, as given in

$$|\mathbf{1}\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ \cdot \\ 0 \end{pmatrix}; \quad |\mathbf{2}\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ \cdot \\ 0 \end{pmatrix}; \ldots; \quad |\mathbf{i}\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \ldots; \quad |\mathbf{N}\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 1 \end{pmatrix} \tag{1.50}$$

The bra $\langle\mathbf{A}|$ is defined as the transpose conjugate (also adjoint) $|\mathbf{A}\rangle^*$ of the ket $|\mathbf{A}\rangle$. Therefore, the bra is a row vector, and its elements are the conjugates of the ket elements.

To formulate the bra of $|\mathbf{A}\rangle$, at first the ket $|\mathbf{A}\rangle$ must be transposed; then, its complex elements are conjugated into the bra elements.

$$\begin{aligned} |\mathbf{A}\rangle &\equiv \begin{pmatrix} \alpha_1 + \mathrm{j}\beta_1 \\ \alpha_2 + \mathrm{j}\beta_2 \\ \ldots \\ \alpha_N + \mathrm{j}\beta_N \end{pmatrix} \\ &\Rightarrow |\mathbf{A}^T\rangle = [(\alpha_1 + \mathrm{j}\beta_1) \quad (\alpha_2 + \mathrm{j}\beta_2) \quad \ldots \quad (\alpha_N + \mathrm{j}\beta_N)] \\ &\Rightarrow |\mathbf{A}\rangle^* = [(\alpha_1 - \mathrm{j}\beta_1) \quad (\alpha_2 - \mathrm{j}\beta_2) \quad \ldots \quad (\alpha_N - \mathrm{j}\beta_N)] \equiv \langle\mathbf{A}| \end{aligned} \tag{1.51}$$

The ket vector $|\mathbf{A}\rangle^*$ is called the transpose conjugate (adjoint) of the ket $|\mathbf{A}\rangle$ and equals the bra $\langle\mathbf{A}|$.

Thus, the bra $\langle \mathbf{A}|$ can be written in the bra bases

$$\langle \mathbf{A}| \equiv |\mathbf{A}\rangle^* = \sum_{j=1}^{N} \langle \mathbf{j}| \cdot a_j^*;\; a_j^* \equiv \left(\alpha_j - \mathrm{j}\beta_j\right) \in \mathbf{C} \tag{1.52}$$

Analogously, the bra bases result from Eq. (1.50).

$$\begin{aligned} &\langle \mathbf{1}| \equiv [1 \quad 0 \quad \cdot \quad 0]; \quad \langle \mathbf{2}| \equiv [0 \quad 1 \quad \cdot \quad 0]; \\ &\langle \mathbf{j}| \equiv [0 \quad 0 \quad 1 \quad 0]; \quad \langle \mathbf{N}| \equiv [0 \quad 0 \quad 0 \quad 1] \end{aligned} \tag{1.53}$$

Due to orthonormality, the product of bra and ket is a scalar and obviously equals the Kronecker delta.

$$\langle \mathbf{i}| \cdot |\mathbf{j}\rangle \equiv \langle \mathbf{i} \mid \mathbf{j}\rangle = \delta_i^j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{1.54}$$

The combined symbol $\langle \mathbf{i}|\mathbf{j}\rangle$ of the bra $\langle \mathbf{i}|$ and ket $|\mathbf{j}\rangle$ in Eq. (1.54) is defined as the inner product (scalar product) of two kets $|\mathbf{i}\rangle$ and $|\mathbf{j}\rangle$.

1.6.2 Gram–Schmidt Scheme of Basis Orthonormalization

The basis $\{|g_i\rangle\}$ is non-orthonormal in the curvilinear coordinates in the space $\mathbf{R}^3$. Using the Gram–Schmidt scheme (Shankar 1980; Griffiths 2005), the orthonormal bases $(|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, |\mathbf{e}_3\rangle)$ are created from the non-orthogonal bases $(|\mathbf{g}_1\rangle, |\mathbf{g}_2\rangle, |\mathbf{g}_3\rangle)$. The orthonormalization procedure of the basis is discussed in Appendix E.2.6.

The first orthonormal ket basis is generated by

$$|\mathbf{e}_1\rangle \equiv |\mathbf{1}\rangle = \frac{|\mathbf{g}_1\rangle}{|\mathbf{g}_1|}$$

The second orthonormal ket basis results from

$$|\mathbf{e}_2\rangle \equiv |\mathbf{2}\rangle = \frac{|\mathbf{g}_2\rangle - \langle \mathbf{e}_1 \mid \mathbf{g}_2\rangle \cdot |\mathbf{e}_1\rangle}{\left||\mathbf{g}_2\rangle - \langle \mathbf{e}_1 \mid \mathbf{g}_2\rangle \cdot |\mathbf{e}_1\rangle\right|}$$

The third orthonormal ket basis is similarly calculated in

$$|\mathbf{e}_3\rangle \equiv |\mathbf{3}\rangle = \frac{|\mathbf{g}_3\rangle - \langle \mathbf{e}_1 \mid \mathbf{g}_3\rangle \cdot |\mathbf{e}_1\rangle - \langle \mathbf{e}_2 \mid \mathbf{g}_3\rangle \cdot |\mathbf{e}_2\rangle}{\left||\mathbf{g}_3\rangle - \langle \mathbf{e}_1 \mid \mathbf{g}_3\rangle \cdot |\mathbf{e}_1\rangle - \langle \mathbf{e}_2 \mid \mathbf{g}_3\rangle \cdot |\mathbf{e}_2\rangle\right|}$$

Generally, the orthonormal ket basis $\left|\mathbf{e}_j\right\rangle$ can be rewritten in the N-dimensional space.

$$\left|\mathbf{e}_j\right\rangle \equiv |\mathbf{j}\rangle = \frac{\left|\mathbf{g}_j\right\rangle - \sum_{i=1}^{j-1}\left\langle \mathbf{e}_i \mid \mathbf{g}_j\right\rangle \cdot |\mathbf{e}_i\rangle}{\left|\left|\mathbf{g}_j\right\rangle - \sum_{i=1}^{j-1}\left\langle \mathbf{e}_i \mid \mathbf{g}_j\right\rangle \cdot |\mathbf{e}_i\rangle\right|} \quad \text{for} \quad j = 1, 2, \ldots, N \tag{1.55}$$

Using the Gram–Schmidt procedure, the ket orthonormal bases of $\{|\mathbf{1}\rangle, |\mathbf{2}\rangle, \ldots, \{|\mathbf{N}\rangle\}$ in the coordinates $(u^1,\ldots, u^N)$ are generated from any non-orthonormal bases, as given in Eq. (1.50). The bra orthonormal bases of $\{\langle\mathbf{1}|, \langle\mathbf{2}|,\ldots, \langle\mathbf{N}|\}$ in the coordinates $(u^1,\ldots, u^N)$ are the adjoint of the ket orthonormal bases.

1.6.3 Cauchy–Schwarz and Triangle Inequalities

The Cauchy–Schwarz and triangle inequalities immediately apply to the Bra and Ket notation:

1. Cauchy–Schwarz Inequality

The well-known Cauchy–Schwarz inequality provides the relation between the inner product of two kets and the ket norms.

$$\langle \mathbf{A} \mid \mathbf{B}\rangle \le ||\mathbf{A}\rangle| \cdot ||\mathbf{B}\rangle|; \quad |\mathbf{A}\rangle, \quad |\mathbf{B}\rangle \in \mathbf{V}^N \tag{1.56}$$

2. Triangle Inequality

The triangle inequality formulates the inequality between the sum of two kets and the ket norms.

$$||\mathbf{A}\rangle + |\mathbf{B}\rangle| \le ||\mathbf{A}\rangle| + ||\mathbf{B}\rangle|; \quad |\mathbf{A}\rangle, \quad |\mathbf{B}\rangle \in \mathbf{V}^N. \tag{1.57}$$

1.6.4 Computing Ket and Bra Components

The component of a ket results from multiplying the ket by a bra basis according to Eqs. (1.49 and 1.54):

$$\begin{aligned} \langle\mathbf{j}| \cdot |\mathbf{A}\rangle \equiv \langle \mathbf{j} \mid \mathbf{A}\rangle &= \sum_{i=1}^{N} \langle\mathbf{j}|.(a_i|\mathbf{i}\rangle) \\ &= \sum_{i=1}^{N} \langle \mathbf{j} \mid \mathbf{i}\rangle \cdot a_i = \sum_{i=1}^{N} \delta_j^i \cdot a_i \\ &= a_j \equiv (\alpha_j + \mathrm{j}\beta_j) \end{aligned} \tag{1.58}$$

Equation (1.58) indicates that the ket component in the orthogonal bases equals the scalar product between the ket and its relating basis.

Similarly, the bra component can be computed by multiplying the bra by a ket basis.

$$\begin{aligned}\langle \mathbf{A}| \cdot |\mathbf{j}\rangle \equiv \langle \mathbf{A} \mid \mathbf{j}\rangle &= \sum_{i=1}^{N} (\langle \mathbf{i}|a_i^*) \cdot |\mathbf{j}\rangle \\ &= \sum_{i=1}^{N} \langle \mathbf{i} \mid \mathbf{j}\rangle \cdot a_i^* = \sum_{i=1}^{N} \delta_i^j \cdot a_i^* \\ &= a_j^* \equiv (\alpha_j - \mathrm{j}\beta_j)\end{aligned} \tag{1.59}$$

It is straightforward that the bra component is equal to the conjugate of the relating ket component a_j, as given in Eq. (1.52).

1.6.5 Inner Product of Two Kets

The inner product of two kets $|\mathbf{A}\rangle$ and $|\mathbf{B}\rangle$ is defined by

$$\begin{aligned}\langle \mathbf{A} \mid \mathbf{B}\rangle &= \left(\sum_{i=1}^{N} \langle \mathbf{i}|a_i^*\right) \cdot \left(\sum_{j=1}^{N} b_j|\mathbf{j}\rangle\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i^* b_j \langle \mathbf{i} \mid \mathbf{j}\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i^* b_j \delta_i^j \\ &= \sum_{i=1}^{N} a_i^* b_i\end{aligned} \tag{1.60a}$$

It is obvious that the inner product of two kets is a complex number according to Eqs. (1.59 and 1.60a).

In the case of $|\mathbf{A}\rangle = |\mathbf{B}\rangle$, the inner product in Eq. (1.60a) becomes

$$\begin{aligned}\langle \mathbf{A} \mid \mathbf{A}\rangle &= \left(\sum_{i=1}^{N} \langle \mathbf{i}|a_i^*\right) \cdot \left(\sum_{j=1}^{N} a_j|\mathbf{j}\rangle\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i^* a_j \langle \mathbf{i} \mid \mathbf{j}\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i^* a_j \delta_i^j \\ &= \sum_{i=1}^{N} a_i^* a_i = \sum_{i=1}^{N} (\alpha_i^2 + \beta_i^2) = ||\mathbf{A}\rangle|^2\end{aligned} \tag{1.60b}$$

Thus, the norm (length) of the ket $|\mathbf{A}\rangle$ is given in

$$||\mathbf{A}\rangle| = \sqrt{\langle \mathbf{A} \mid \mathbf{A}\rangle} \tag{1.60c}$$

The inner product in Eq. (1.60a) can be rewritten using Eq. (1.52).

$$\begin{aligned}\langle \mathbf{A} \mid \mathbf{B}\rangle &= \sum_{i=1}^{N} a_i^* b_i = [a_1^* \quad a_2^* \quad a_i^* \quad a_N^*].\begin{bmatrix} b_1 \\ b_2 \\ b_i \\ b_N \end{bmatrix} \\ &= |\mathbf{A}\rangle^* \cdot |\mathbf{B}\rangle = \langle \mathbf{A}| \cdot |\mathbf{B}\rangle \equiv \langle \mathbf{A} \mid \mathbf{B}\rangle \end{aligned} \tag{1.61}$$

Similarly, the inner product of two kets$|\mathbf{B}\rangle$ and $|\mathbf{A}\rangle$ can be calculated as follows:

$$\begin{aligned}\langle \mathbf{B} \mid \mathbf{A}\rangle &= \left(\sum_{j=1}^{N} \langle \mathbf{j}| b_j^*\right) \cdot \left(\sum_{i=1}^{N} a_i |\mathbf{i}\rangle\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} b_j^* a_i \langle \mathbf{j} \mid \mathbf{i}\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} b_j^* a_i \delta_j^i \\ &= \sum_{i=1}^{N} b_i^* a_i \end{aligned} \tag{1.62}$$

By conjugating Eq. (1.62), one obtains the transpose conjugate of $\langle \mathbf{B}|\mathbf{A}\rangle$:

$$\begin{aligned}\langle \mathbf{B} \mid \mathbf{A}\rangle^* &= \left(\sum_{i=1}^{N} b_i^* a_i\right)^* \\ &= \sum_{i=1}^{N} a_i^* \left(b_i^*\right)^* = \sum_{i=1}^{N} a_i^* b_i \\ &= \langle \mathbf{A} \mid \mathbf{B}\rangle \end{aligned} \tag{1.63}$$

Thus, the inner product is skew-symmetric (antisymmetric) contrary to the inner product of two regular vectors.

Some properties of the inner product (scalar product) are valid:

Skew symmetry: $\langle \mathbf{A}|\mathbf{B}\rangle = \langle \mathbf{B}|\mathbf{A}\rangle^*$;
Positive definiteness: $\langle \mathbf{A}|\mathbf{A}\rangle = ||\mathbf{A}\rangle|^2 \geq 0$;
Distributive property: $\langle \mathbf{A}|(\alpha\mathbf{B} + \beta\mathbf{C})\rangle = \alpha\langle \mathbf{A}|\mathbf{B}\rangle + \beta\langle \mathbf{A}|\mathbf{C}\rangle$ for α, $\beta \in \mathbf{C}$.

Furthermore, the linear adjoint operator has the following properties:

1. $(\alpha\beta\gamma)^* = [\alpha(\beta\gamma)]^* = (\beta\gamma)^*\alpha^* = \gamma^*\beta^*\alpha^*$ for α, β, $\gamma \in \mathbf{C}$

Note that the product order is changed in the adjoint operation of the scalar product.

2. $(\alpha|\mathbf{A}\rangle)^* = |\alpha\mathbf{A}\rangle^* = |\mathbf{A}\rangle^*\alpha^* = \langle\mathbf{A}|\,\alpha^*$ for $\alpha \in \mathbf{C}$
3. $(\langle\mathbf{A}|\alpha)^* = \alpha^*\langle\mathbf{A}|^* = \alpha^*|\mathbf{A}\rangle$ for $\alpha \in \mathbf{C}$
4. $\langle\mathbf{A}|\alpha^{**} = \langle\mathbf{A}|\alpha$ for $\alpha \in \mathbf{C}$
5. $\langle\mathbf{B}|\mathbf{A}\rangle^* = |\mathbf{A}\rangle^*\cdot\langle\mathbf{B}|^* = \langle\mathbf{A}|\cdot|\mathbf{B}\rangle \equiv \langle\mathbf{A}|\mathbf{B}\rangle$: skew-symmetric (antisymmetric)
6. $\langle\mathbf{A}|\alpha^*|\mathbf{B}\rangle^* = |\mathbf{B}\rangle^*\cdot\alpha\langle\mathbf{A}|^* = \langle\mathbf{B}|\cdot\alpha\cdot|\mathbf{A}\rangle \equiv \langle\mathbf{B}|\alpha|\mathbf{A}\rangle$ for $\alpha \in \mathbf{C}$
7. $(|\mathbf{A}\rangle\langle\mathbf{B}|)^* = \langle\mathbf{B}|^*\cdot|\mathbf{A}\rangle^* = |\mathbf{B}\rangle\langle\mathbf{A}|$: the outer product of ket and bra.

1.6.6 Outer Product of Bra and Ket

The outer product of the ket $|\mathbf{A}\rangle$ and the bra $\langle\mathbf{B}|$ is defined as

$$\begin{aligned} |\mathbf{A}\rangle\langle\mathbf{B}| &= \left(\sum_{i=1}^{N} a_i|\mathbf{i}\rangle\right)\cdot\left(\sum_{j=1}^{N}\langle\mathbf{j}|b_j^*\right) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_i b_j^*\,|\mathbf{i}\rangle\,\langle\mathbf{j}| \end{aligned} \tag{1.64}$$

where the product term $|\mathbf{i}\rangle\langle\mathbf{j}|$ is called the outer product of the bases $|\mathbf{i}\rangle$ and $\langle\mathbf{j}|$.

Contrary to the inner product resulting a scalar of (1×1) matrix $\in \mathbf{V}$, the outer product is an operator of $(N \times N)$ matrix $\in \mathbf{V}^{N\times N}$ because the ket is an $(N \times 1)$ column vector $\in \mathbf{V}^N$ and the bra is a $(1 \times N)$ row vector $\in \mathbf{V}^N$.

Now, the ket $|\mathbf{A}\rangle$ can be expressed in ket bases:

$$|\mathbf{A}\rangle = \sum_{i=1}^{N}|\mathbf{i}\rangle\,a_i \tag{1.65}$$

According to Eq. (1.58), the ket component is

$$a_i = \langle\mathbf{i}\mid\mathbf{A}\rangle$$

Substituting a_i into Eq. (1.65), one obtains the ket

$$|\mathbf{A}\rangle = \sum_{i=1}^{N}|\mathbf{i}\rangle\,\langle\mathbf{i}\mid\mathbf{A}\rangle \equiv \sum_{i=1}^{N}\mathbf{I}_i|\mathbf{A}\rangle \tag{1.66}$$

where $\mathbf{I}_i$ is the projection operator (outer product) according to (Shankar 1980), as defined by

$$\mathbf{I}_{\mathrm{i}} \equiv |\mathbf{i}\rangle\langle\mathbf{i}| = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.67}$$

The element I_{ii} of the projection operator (matrix) is 1 at the ith row and jth column, as shown in Eq. (1.67); otherwise, other elements are equal to zero.

Obviously, the sum of all projection operators is the identity matrix.

$$\mathbf{I} \equiv \sum_{i=1}^{N} \mathbf{I}_i = \sum_{i=1}^{\mathrm{N}} |\mathbf{i}\rangle\, \langle\mathbf{i}| \tag{1.68}$$

According to Eq. (1.66), the identity property of the ket is proved by

$$|\mathbf{A}\rangle = \sum_{i=1}^{N} \mathbf{I}_i|\mathbf{A}\rangle = \mathbf{I}|\mathbf{A}\rangle. \tag{1.69}$$

1.6.7 Ket and Bra Projection Components on the Bases

The projection component of the ket $|\mathbf{A}\rangle$ on the basis $|\mathbf{i}\rangle$ can be calculated as

$$\begin{aligned} |\mathbf{A}\rangle_i &= \mathbf{I}_i|\mathbf{A}\rangle \\ &= |\mathbf{i}\rangle\, \langle\mathbf{i}| \cdot |\mathbf{A}\rangle = |\mathbf{i}\rangle\, \langle\mathbf{i} \mid \mathbf{A}\rangle \\ &= |\mathbf{i}\rangle a_i \end{aligned} \tag{1.70}$$

Thus, the ket can be expressed, as shown in Eq. (1.49):

$$|\mathbf{A}\rangle = \sum_{i=1}^{N} |\mathbf{A}\rangle_i = \sum_{i=1}^{N} |\mathbf{i}\rangle a_i \tag{1.71}$$

Similarly, the projection component of the bra $\langle\mathbf{A}|$ on the basis $\langle\mathbf{i}|$ computes as

$$\begin{aligned} \langle\mathbf{A}|_i &= \langle\mathbf{A}|\mathbf{I}_i \\ &= \langle\mathbf{A}| \cdot |\mathbf{i}\rangle\, \langle\mathbf{i}| = \langle\mathbf{A} \mid \mathbf{i}\rangle\, \langle\mathbf{i}| \\ &= \langle\mathbf{i}|a_i^* \end{aligned} \tag{1.72}$$

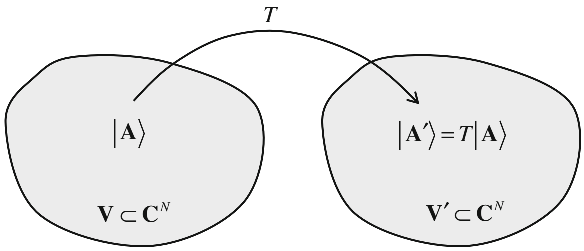

Fig. 1.6 Linear transformation T of a ket $|\mathbf{A}\rangle$

The bra can be expressed in its projection components, as given in Eq. (1.52).

$$\begin{aligned}\langle \mathbf{A}| &= \sum_{i=1}^{N} \langle \mathbf{A}|_i = \sum_{i=1}^{N} \langle \mathbf{i}|a_i^* \\ &= |\mathbf{A}\rangle^* = \left(\sum_{i=1}^{N} |\mathbf{i}\rangle a_i \right)^* = \sum_{i=1}^{N} a_i^* \langle \mathbf{i}|.\end{aligned} \tag{1.73}$$

1.6.8 Linear Transformation of Kets

We consider the complex vector spaces $\mathbf{V}$ and $\mathbf{V}'$. Each of them belongs to the finite N-dimensional complex space $\mathbf{C}^N$. The linear transformation T maps the ket $|\mathbf{A}\rangle$ in $\mathbf{V}$ into the image ket $|\mathbf{A}'\rangle$ in $\mathbf{V}'$, as shown in Fig. 1.6.

The image ket $|\mathbf{A}'\rangle$ can be written in the bases $|\mathbf{i}\rangle$ (Shankar 1980; Griffiths 2005) as

$$\begin{aligned}&T : |\mathbf{A}\rangle \to |\mathbf{A}'\rangle = T|\mathbf{A}\rangle \\ &|\mathbf{A}'\rangle = T \sum_{i=1}^{N} |\mathbf{i}\rangle a_i = \sum_{i=1}^{N} T|\mathbf{i}\rangle a_i\end{aligned} \tag{1.74}$$

In this case, the ket basis $|\mathbf{i}\rangle$ is also mapped into the image ket basis $|\mathbf{i}'\rangle$. The transformation operator T for the basis can be written as

$$T : |\mathbf{i}\rangle \to |\mathbf{i}'\rangle \Rightarrow |\mathbf{i}'\rangle = T|\mathbf{i}\rangle \tag{1.75}$$

The image ket basis $|\mathbf{i}'\rangle$ is formulated as a linear combination of the old ket bases $\{|\mathbf{i}\rangle, \mathbf{j} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{N})\}$.

$$|\mathbf{i}'\rangle = T|\mathbf{i}\rangle = \sum_{j=1}^{N} T_{ji}|\mathbf{j}\rangle; \quad \mathbf{i} = 1, 2, \ldots, N \tag{1.76}$$

where the operator element T_{ji} is in the jth row and ith column of the transformation matrix $\mathbf{T}$ of the transformation operator T.

Multiplying both sides of Eq. (1.76) by the bra basis $\langle \mathbf{k}|$, one obtains

$$\begin{aligned}\langle \mathbf{k} \mid \mathbf{i}' \rangle = \langle \mathbf{k}|T|\mathbf{i}\rangle = \langle \mathbf{k}| \cdot \sum_{j=1}^{N} T_{ji}|\mathbf{j}\rangle \\ = \sum_{j=1}^{N} T_{ji}\langle \mathbf{k} \mid \mathbf{j}\rangle = \sum_{j=1}^{N} T_{ji}\delta_k^j = T_{ki}\end{aligned} \tag{1.77}$$

Thus, the operator element results from Eq. (1.77):

$$T_{ki} = \langle \mathbf{k} \mid \mathbf{i}' \rangle = \langle \mathbf{k}|T|\mathbf{i}\rangle \tag{1.78}$$

Substituting Eq. (1.76) into Eq. (1.74), one obtains the image ket.

$$\begin{aligned}|\mathbf{A}'\rangle &= T|\mathbf{A}\rangle = T\sum_{i=1}^{N} |\mathbf{i}\rangle a_i \\ &= \sum_{i=1}^{N} T|\mathbf{i}\rangle a_i = \sum_{i=1}^{N} \left(\sum_{j=1}^{N} T_{ji}|\mathbf{j}\rangle \right) a_i \\ &= \sum_{j=1}^{N} \left(\sum_{i=1}^{N} T_{ji}a_i \right) |\mathbf{j}\rangle \equiv \sum_{j=1}^{N} a_j'|\mathbf{j}\rangle\end{aligned} \tag{1.79}$$

The component of the image ket in the basis $|\mathbf{j}\rangle$ is given from Eqs. (1.78 and 1.79).

$$\begin{aligned}a_j' = \sum_{i=1}^{N} T_{ji}a_i = \sum_{i=1}^{N} \langle \mathbf{j}|T|\mathbf{i}\rangle\, a_i \\ \Leftrightarrow |\mathbf{A}'\rangle = \mathbf{T}_{N\times N}|\mathbf{A}\rangle\end{aligned} \tag{1.80}$$

The image ket in Eq. (1.80) can be rewritten in the formulation matrix $T_{N\times N}$.

$$\begin{bmatrix} a_1' \\ a_2' \\ \cdot \\ a_j' \\ a_N' \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & \cdot & T_{1i} & T_{1N} \\ T_{21} & T_{22} & \cdot & T_{2i} & T_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ T_{j1} & \cdot & \cdot & T_{ji} & T_{jN} \\ T_{N1} & T_{N2} & \cdot & T_{Ni} & T_{NN} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ a_i \\ a_N \end{bmatrix} \tag{1.81}$$

where the matrix element T_{ji} is computed by the ket transformation T, as given in Eq. (1.78).

$$T_{ji} = \langle \mathbf{j}|T|\mathbf{i}\rangle.$$

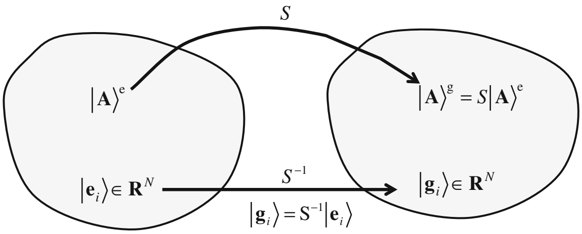

Fig. 1.7 Coordinate transformation of bases

1.6.9 Coordinate Transformations

The ket basis $|\mathbf{e}_i\rangle$ is transformed into the new ket basis $|\mathbf{g}_i\rangle$ by the transformation S^{-1}, as shown in Fig. 1.7. The transformed ket basis can be written as (Griffiths 2005).

$$\begin{aligned} S^{-1}: \ |\mathbf{e}_i\rangle \to |\mathbf{g}_i\rangle \Rightarrow |\mathbf{g}_i\rangle = S^{-1}|\mathbf{e}_i\rangle \\ \Leftrightarrow S: \ |\mathbf{g}_i\rangle \to |\mathbf{e}_i\rangle \Rightarrow |\mathbf{e}_i\rangle = S|\mathbf{g}_i\rangle \end{aligned} \tag{1.82}$$

Analogous to the ket transformation, the old ket basis can be written in a linear combination of the new bases:

$$|\mathbf{e}_i\rangle = \sum_{j=1}^{\mathrm{N}} |\mathbf{g}_j\rangle S_{ji} \quad \text{for} \quad i = 1, 2, \ldots, N \tag{1.83}$$

in which S_{ji} is the matrix element of the transformation matrix $\mathbf{S}$.

Multiplying both sides of Eq. (1.83) by the bra basis $\langle \mathbf{g}_k|$ one obtains the matrix element S_{ki}.

$$\begin{aligned} \langle \mathbf{g}_k \mid \mathbf{e}_i\rangle &= \langle \mathbf{g}_k| \cdot \sum_{j=1}^{N} |\mathbf{g}_j\rangle S_{ji} = \sum_{j=1}^{N} \langle \mathbf{g}_k \mid \mathbf{g}_j\rangle S_{ji} \\ &= \sum_{j=1}^{N} \delta_k^j S_{ji} = S_{ki} \end{aligned} \tag{1.84}$$

Thus, using Eq. (1.82) it gives

$$S_{ki} = \langle \mathbf{g}_k \mid \mathbf{e}_i\rangle = \langle \mathbf{g}_k|S|\mathbf{g}_i\rangle \tag{1.85}$$

An arbitrary ket $|\mathbf{A}\rangle$ can be expressed linearly in terms of the old ket basis:

$$|\mathbf{A}\rangle = \sum_{i=1}^{N} |\mathbf{e}_i\rangle \, a_i^e \tag{1.86}$$

where a_i^e is the ket component in the old basis $|\mathbf{e}_i\rangle$.

Substituting Eq. (1.83) into Eq. (1.86), one obtains

$$\begin{aligned} |\mathbf{A}\rangle &= \sum_{i=1}^{N} |\mathbf{e}_i\rangle\, a_i^e = \sum_{i=1}^{N} \left(\sum_{k=1}^{N} |\mathbf{g}_k\rangle S_{ki} \right) a_i^e \\ &= \sum_{k=1}^{N} \left(\sum_{i=1}^{N} S_{ki} a_i^e \right) \cdot |\mathbf{g}_k\rangle \equiv \sum_{k=1}^{N} a_k^g |\mathbf{g}_k\rangle \end{aligned} \tag{1.87}$$

Therefore, the ket component in the transformed basis $|\mathbf{g}_k\rangle$ can be calculated by Eq. (1.87).

$$a_k^g = \sum_{i=1}^{N} S_{ki} a_i^e = \sum_{i=1}^{N} \langle \mathbf{g}_k | S | \mathbf{g}_i \rangle\, a_i^e \Leftrightarrow |\mathbf{A}\rangle^{\mathrm{g}} = S |\mathbf{A}\rangle^{\mathrm{e}} \tag{1.88}$$

The transformed ket in Eq. (1.88) can be rewritten in the matrix formulation:

$$\begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ a_k \\ a_N \end{bmatrix}^g = \begin{bmatrix} S_{11} & S_{12} & \cdot & S_{1i} & S_{1N} \\ S_{21} & S_{22} & \cdot & S_{2i} & S_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ S_{k1} & \cdot & \cdot & S_{ki} & S_{kN} \\ S_{N1} & S_{N2} & \cdot & S_{Ni} & S_{NN} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ a_i \\ a_N \end{bmatrix}^e \tag{1.89}$$

where the matrix element is given in Eq. (1.89):

$$S_{ki} = \langle \mathbf{g}_k \mid \mathbf{e}_i \rangle = \langle \mathbf{g}_k | S | \mathbf{g}_i \rangle \tag{1.90}$$

The components of the transformed ket a_k^g in the new basis $|\mathbf{g}_k\rangle$ are derived from Eq. (1.89).

In the following section, a combined transformation of kets consisting of three transformations is carried out (Griffiths 2005; Longair 2013), as shown in Fig. 1.8:

1. Basis transformation S^{-1} from $|\mathbf{A}\rangle^g$ in the basis $|\mathbf{g}_i\rangle$ to $|\mathbf{A}\rangle^e$ in the basis $|\mathbf{e}_i\rangle$;
2. Ket transformation T from $|\mathbf{A}\rangle^e$ to $|\mathbf{A}\rangle^e$ in the basis $|\mathbf{e}_i\rangle$;
3. Basis transformation S from $|\mathbf{A}'\rangle^e$ in the basis $|\mathbf{e}_i\rangle$ to $|\mathbf{A}'\rangle^g$ in the basis $|\mathbf{g}_i\rangle$.

The first transformation yields the first transformed ket:

$$|\mathbf{A}\rangle^e = S^{-1} |\mathbf{A}\rangle^g \tag{1.91}$$

The second transformation yields the second transformed ket:

$$|\mathbf{A}'\rangle^e = T |\mathbf{A}\rangle^e \tag{1.92}$$

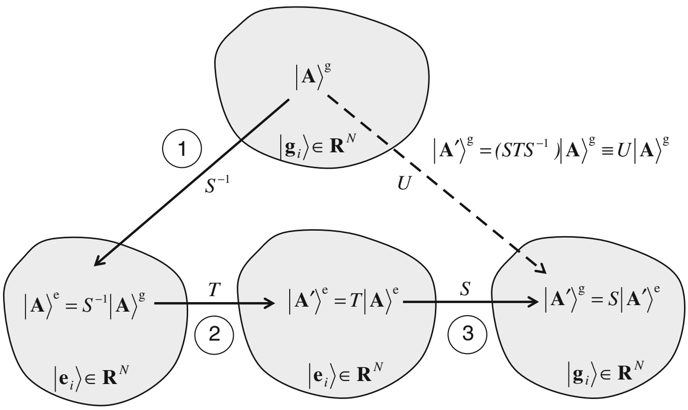

Fig. 1.8 The combined transformation of kets

The third transformation leads to the third transformed ket:

$$|\mathbf{A}'\rangle^{\mathrm{g}} = S|\mathbf{A}'\rangle^{\mathrm{e}} \tag{1.93}$$

Finally, the combined transformed ket of three transformations results from Eqs. (1.91, 1.92, and 1.93).

$$|\mathbf{A}'\rangle^{\mathrm{g}} = S|\mathbf{A}'\rangle^{\mathrm{e}} = ST|\mathbf{A}\rangle^{\mathrm{e}} = (STS^{-1})|\mathbf{A}\rangle^{\mathrm{g}} \equiv U|\mathbf{A}\rangle^{\mathrm{g}} \tag{1.94}$$

where the combined transformation U is defined as (STS^{-1}).

The component of the product of many operators is computed (Shankar 1980; Griffiths 2005) according to Eq. (1.78).

$$\begin{aligned} U_{ij} &= \langle \mathbf{i}|U|\mathbf{j}\rangle = \langle \mathbf{i}|(STS^{-1})|\mathbf{j}\rangle \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} \langle \mathbf{i}|S|\mathbf{k}\rangle \cdot \langle \mathbf{k}|T|\mathbf{l}\rangle \cdot \langle \mathbf{l}|S^{-1}|\mathbf{j}\rangle \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} S_{ik}T_{kl}S_{lj}^{-1} \end{aligned} \tag{1.95}$$

The ket transformed component of $|\mathbf{A}'\rangle^{g}$ can be obtained from Eqs. (1.94 and 1.95).

$$\begin{aligned} &|\mathbf{A}'\rangle^{g} = U|\mathbf{A}\rangle^{g} \\ &\Leftrightarrow a_i'^{g} = \sum_{j=1}^{N} U_{ij}a_j^{g} = \sum_{j=1}^{N}\left(\sum_{k=1}^{N}\sum_{l=1}^{N} S_{ik}T_{kl}S_{lj}^{-1}\right) a_j^{g} \end{aligned} \tag{1.96}$$

The transformed ket in Eq. (1.96) can be rewritten in the formulation matrix $\mathbf{T}_{N\times N}$

$$\begin{bmatrix} a'_1 \\ a'_2 \\ \cdot \\ a'_i \\ a'_N \end{bmatrix}^g = \begin{bmatrix} U_{11} & U_{12} & \cdot & U_{1i} & U_{1N} \\ U_{21} & U_{22} & \cdot & U_{2i} & U_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ U_{i1} & \cdot & \cdot & U_{ij} & U_{iN} \\ U_{N1} & U_{N2} & \cdot & U_{Ni} & U_{NN} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ a_j \\ a_N \end{bmatrix}^g . \tag{1.97}$$

1.6.10 Hermitian Transformation

Hermitian transformation plays a key role in eigenvalue problems using in quantum mechanics. The adjoint $T^\dagger$ (spoken T dagger) of a matrix $\mathbf{T}$ is defined as Hermitian if it equals the transpose conjugate T^* (i.e., $\mathbf{T}^\dagger \equiv \mathbf{T}^*$). In this case, the Hermitian transformation $T^\dagger$ is also equal to the linear transformation T of the transformation matrix T ($T^\dagger = T$).

An arbitrary ket $|\mathbf{B}\rangle$ can be transformed by the linear operator T into a ket:

$$T: \ |\mathbf{B}\rangle \in \mathbf{R}^N \to T|\mathbf{B}\rangle \in \mathbf{R}^N \tag{1.98}$$

The inner product of the ket $|\mathbf{A}\rangle$ and transformed ket $T|\mathbf{B}\rangle$ can be written as (Griffiths 2005):

$$\begin{aligned} \langle \mathbf{A} \mid T\mathbf{B}\rangle &= \mathbf{A}^* \cdot \mathbf{TB} = (\mathbf{A}^*\mathbf{T}) \cdot \mathbf{B} = (\mathbf{T}^*\mathbf{A})^* \cdot \mathbf{B} \\ &\equiv (\mathbf{T}^\dagger\mathbf{A})^* \cdot \mathbf{B} = \left\langle T^\dagger\mathbf{A} \,\middle|\, \mathbf{B}\right\rangle \end{aligned} \tag{1.99}$$

This result shows that the inner product between the transformed ket $|T^\dagger\mathbf{A}\rangle$ and ket $|\mathbf{B}\rangle$ is the same inner product of the ket $|\mathbf{A}\rangle$ and transformed ket $T|\mathbf{B}\rangle$. As a rule of thumb, the inner product does not change when moving the transformation operator T from the second ket into the first bra and changing T into $T^\dagger$.

There are some properties of the inner product with a complex number α and its conjugate α^*:

$$\begin{aligned} \langle \mathbf{A} \mid \alpha\mathbf{B}\rangle &= \alpha\langle \mathbf{A} \mid \mathbf{B}\rangle = \langle \alpha^*\mathbf{A} \mid \mathbf{B}\rangle \quad \text{for} \quad \alpha \in \mathbf{C} \\ \langle \alpha\mathbf{A} \mid \mathbf{B}\rangle &= \alpha^*\langle \mathbf{A} \mid \mathbf{B}\rangle = \langle \mathbf{A} \mid \alpha^*\mathbf{B}\rangle \quad \text{for} \quad \alpha \in \mathbf{C} \end{aligned} \tag{1.100}$$

The eigenvalue problem derives from the characteristic equation:

$$T|\mathbf{A}\rangle = \lambda|\mathbf{A}\rangle; \ |\mathbf{A}\rangle \neq \mathbf{0}, \quad \text{for} \quad \lambda \in \mathbf{C} \tag{1.101}$$

The inner product between the ket $|\mathbf{A}\rangle$ and its transformed ket is given by

$$\langle \mathbf{A} \mid T\mathbf{A}\rangle = \langle \mathbf{A} \mid \lambda\mathbf{A}\rangle = \lambda\langle \mathbf{A} \mid \mathbf{A}\rangle \tag{1.102}$$

Some characteristics of the eigenvalue problem are discussed in the following section (Shankar 1980; Griffiths 2005):

1. Eigenvalue of the Hermitian transformation is a real number.

According to Eq. (1.99), the inner product in Eq. (1.102) with the Hermitian transformation $T^\dagger$ (=T) becomes

$$\begin{aligned} \langle \mathbf{A} \mid T\mathbf{A}\rangle &= \left\langle T^\dagger \mathbf{A} \,\middle|\, \mathbf{A}\right\rangle \equiv \langle T\mathbf{A} \mid \mathbf{A}\rangle \\ &= \langle \lambda\mathbf{A} \mid \mathbf{A}\rangle = \lambda^*\langle \mathbf{A} \mid \mathbf{A}\rangle \end{aligned} \tag{1.103}$$

Comparing Eqs. (1.102) and (1.103), one obtains

$$\lambda^* = \lambda \tag{1.104}$$

This result proves that the eigenvalue λ must be a real number.

2. Eigenkets of the Hermitian transformation are orthogonal.

Given two eigenkets with their different eigenvalues λ and μ, the eigenvalue problems can be formulated:

$$\begin{aligned} T|\mathbf{A}\rangle &= \lambda|\mathbf{A}\rangle; \quad |\mathbf{A}\rangle \neq \mathbf{0}, \\ T|\mathbf{B}\rangle &= \mu|\mathbf{B}\rangle; \quad |\mathbf{B}\rangle \neq \mathbf{0}, \end{aligned} \tag{1.105}$$

The inner product between the kets $|\mathbf{A}\rangle$ and $T|\mathbf{B}\rangle$ can be given according to Eq. (1.100).

$$\langle \mathbf{A} \mid T\mathbf{B}\rangle = \langle \mathbf{A} \mid \mu\mathbf{B}\rangle = \mu\langle \mathbf{A} \mid \mathbf{B}\rangle \tag{1.106}$$

Similarly, the inner product between the kets $T|\mathbf{A}\rangle$ and $|\mathbf{B}\rangle$ can be rewritten according to Eq. (1.100).

$$\langle T\mathbf{A} \mid \mathbf{B}\rangle = \langle \lambda\mathbf{A} \mid \mathbf{B}\rangle = \lambda^*\langle \mathbf{A} \mid \mathbf{B}\rangle \tag{1.107}$$

Comparing Eqs. (1.106) to (1.107), one obtains the Hermitian transformation $T^\dagger$ (=T) according to Eq. (1.99):

$$\begin{aligned} \langle \mathbf{A} \mid T\mathbf{B}\rangle &= \left\langle T^\dagger \mathbf{A} \,\middle|\, \mathbf{B}\right\rangle = \langle T\mathbf{A} \mid \mathbf{B}\rangle \\ &\Leftrightarrow \mu\langle \mathbf{A} \mid \mathbf{B}\rangle = \lambda^*\langle \mathbf{A} \mid \mathbf{B}\rangle \end{aligned} \tag{1.108}$$

Therefore,

$$(\mu - \lambda^*)\langle \mathbf{A} \mid \mathbf{B} \rangle = 0 \tag{1.109}$$

Because the eigenvalues μ and λ^* are different, the inner product $\langle \mathbf{A}|\mathbf{B}\rangle$ must be zero according to Eq. (1.109). This result indicates that the eigenkets $|\mathbf{A}\rangle$ and $|\mathbf{B}\rangle$ are orthogonal.

3. Hermitian matrix is diagonalizable in the normalized basis.

For the eigenvalue problem in Eq. (1.101), there exists an eigenket (eigenvector) relating to its eigenvalue. Instead of the formulation given in Eq. (1.105), the Hermitian transformation matrix can be easily written in the orthonormal basis $|\mathbf{e}_i\rangle$ for the eigenvalue λ_i:

$$\mathbf{T}|\mathbf{e}_i\rangle = \lambda_i|\mathbf{e}_i\rangle \Leftrightarrow \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & \lambda_N \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \lambda_i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{for} \quad i = 1, 2, \ldots, N \tag{1.110}$$

Therefore, the Hermitian matrix $\mathbf{T}$ is obviously diagonalizable at changing the eigenvector basis $|\mathbf{X}_i\rangle$ into the orthonormal basis $|\mathbf{e}_i\rangle$. In this case, the eigenvalues locate on the main diagonal and other matrix elements are zero in the Hermitian matrix, as shown in Eq. (1.110) (Griffiths 2005).

1.7 Applying Bra and Ket Analysis to Eigenvalue Problems

Many problems in physics and engineering can be formulated similar to

$$|\dot{\mathbf{X}}\rangle = \mathbf{T}|\mathbf{X}\rangle \tag{1.111}$$

The solution of Eq. (1.111) can be assumed as

$$|\mathbf{X}\rangle = |\mathbf{E}\rangle\, e^{\lambda t} \tag{1.112}$$

where $|\mathbf{E}\rangle$ is the eigenvector, λ is the complex eigenvalue, $\lambda = \alpha + j\omega \in \mathbf{C}$ in which ω is the eigenfrequency and α is the growth/decay rate.

Calculating the first derivative of the solution, one obtains

$$|\dot{\mathbf{X}}\rangle = \lambda|\mathbf{E}\rangle\, e^{\lambda t} = \lambda|\mathbf{X}\rangle \tag{1.113}$$

Substituting Eq. (1.113) into Eq. (1.111), the eigenvalue problem is given as

$$\mathbf{T}|\mathbf{X}\rangle = \lambda|\mathbf{X}\rangle \tag{1.114}$$

Equation (1.114) can be rewritten in the matrix form with the identity matrix **I**.

$$(\mathbf{T} - \lambda\mathbf{I})|\mathbf{X}\rangle = |\mathbf{0}\rangle \tag{1.115}$$

For non-trivial solutions of Eq. (1.115), the eigenvalue-related determinant must be zero.

$$\det(\mathbf{T} - \lambda\mathbf{I}) \equiv |\mathbf{T} - \lambda\mathbf{I}| = 0 \tag{1.116}$$

Equation (1.116) is called the characteristic equation whose solutions are the eigenvalues. The characteristic equation is the polynomial of λ^n; n equals two times of the degrees of freedom (DOF) of the system.

$$P(\lambda^n) \equiv a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \tag{1.117}$$

There exists an eigenvector (eigenket) for each eigenvalue. The eigenvector results from Eq. (1.115).

$$\begin{gathered}(\mathbf{T} - \lambda_{\mathrm{i}}\mathbf{I})|\mathbf{X}_{\mathrm{i}}\rangle = (\mathbf{T} - \lambda_{\mathrm{i}}\mathbf{I})|\mathbf{E}_{\mathrm{i}}\rangle e^{\lambda_i t} = |\mathbf{0}\rangle \\ \forall\lambda_i \in \mathbf{C} \Rightarrow (\mathbf{T} - \lambda_{\mathrm{i}}\mathbf{I})|\mathbf{E}_{\mathrm{i}}\rangle = |\mathbf{0}\rangle\end{gathered} \tag{1.118}$$

An example for the eigenvalue problem will be given in the following subsection.

Let the system matrix **T** be given as:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The characteristic equation of the eigenvalues λ yields

$$\begin{aligned}|\mathbf{T} - \lambda\mathbf{I}| &= \begin{vmatrix} (1-\lambda) & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 1 & (1-\lambda) \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & (1-\lambda) \end{vmatrix} = (1-\lambda)[\lambda(\lambda-1)+1] \\ &= (1-\lambda)(\lambda^2 - \lambda + 1) \\ &= (1-\lambda)\left(\lambda - \frac{1+\mathrm{j}\sqrt{3}}{2}\right)\cdot\left(\lambda - \frac{1-\mathrm{j}\sqrt{3}}{2}\right) = 0\end{aligned}$$

Thus, there are three eigenvalues as follows:

$$\lambda_1 = 1;$$
$$\lambda_2 = \frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2};$$
$$\lambda_3 = \frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2}.$$

Using Eq. (1.118), one obtains the eigenvectors of the eigenvalue problem:

- For $\lambda = \lambda_1 = 1$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$\begin{cases} 0x_1 = 0 \rightarrow x_1 \equiv 1 \\ -x_2 - x_3 = 0 \rightarrow x_3 = -x_2 = 0 \\ x_2 = 0 \rightarrow x_2 = 0 \end{cases} \Rightarrow |\mathbf{E}_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- For $\lambda = \lambda_2 = \frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2}$:

$$\begin{bmatrix} \frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\left(\frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2}\right) & -1 \\ 0 & 1 & \frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$\begin{cases} \left(\frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2}\right) x_1 = 0 \rightarrow x_1 = 0 \\ -\left(\frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2}\right) x_2 - x_3 = 0 \rightarrow x_2 \equiv 2\mathrm{j} \\ x_2 + \left(\frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2}\right) x_3 = 0 \rightarrow x_3 = \sqrt{3} - \mathrm{j} \end{cases} \Rightarrow |\mathbf{E}_2\rangle = \begin{pmatrix} 0 \\ 2\mathrm{j} \\ \sqrt{3} - \mathrm{j} \end{pmatrix}$$

- For $\lambda = \lambda_3 = \frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2}$:

$$\begin{bmatrix} \frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & -\left(\frac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2}\right) & -1 \\ 0 & 1 & \frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$\begin{cases} \left(\dfrac{1}{2} + \mathrm{j}\dfrac{\sqrt{3}}{2}\right)x_1 = 0 \rightarrow x_1 = 0 \\ -\left(\dfrac{1}{2} - \mathrm{j}\dfrac{\sqrt{3}}{2}\right)x_2 - x_3 = 0 \rightarrow x_2 \equiv -2\mathrm{j} \\ x_2 + \left(\dfrac{1}{2} + \mathrm{j}\dfrac{\sqrt{3}}{2}\right)x_3 = 0 \rightarrow x_3 = \sqrt{3} + \mathrm{j}. \end{cases} \Rightarrow |\mathbf{E}_3\rangle = \begin{pmatrix} 0 \\ -2\mathrm{j} \\ \sqrt{3} + \mathrm{j} \end{pmatrix}.$$

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Chapter 2
Tensor Analysis

2.1 Introduction to Tensors

Tensors are a powerful mathematical tool that is used in many areas in engineering and physics including general relativity theory, quantum mechanics, statistical thermodynamics, classical mechanics, electrodynamics, solid mechanics, and fluid dynamics. Laws of physics and physical invariants must be independent of any arbitrarily chosen coordinate system. However, the tensor components describing these characteristics heavily depend on the coordinate bases and therefore change as the coordinate system varies in the considered spaces. Before going into detail, we provide less-experienced readers with some examples.

Different tensors are listed in Table 2.1, which can be expressed in differently chosen bases for any curvilinear coordinate. Using Einstein summation convention, the notation can be shortened. Note that Einstein summation convention is only valid for the same indices in the lower and upper positions. The relating contravariant or covariant tensor components can be expressed in the covariant or contravariant bases (cf. Appendix E). The tensor order is determined by the number of the coordinate basis. Thus, the component of a first-order tensor has only one dummy index i relating to a single basis. In the case of a second-order tensor, its component contains two dummy indices i and j relating to double bases. Similarly, the component of an N-order tensor has N dummy indices relating to N bases.

The dummy indices (inner indices) are the repeated indices running from the values from 1 to N in Einstein summation convention. The free index (outer index) can be independently chosen for any value from 1 to N, that is, for any tensor component in the particular coordinate, as shown in the below example. Note that the dimensions of the dummy and free indices must be the same value of the space dimensions.

H. Nguyen-Schäfer and J.-P. Schmidt, *Tensor Analysis and Elementary Differential Geometry for Physicists and Engineers*, Mathematical Engineering 21,
DOI: 10.1007/978-3-662-43444-4_2,

Table 2.1 Tensors in general curvilinear coordinates

Type	Component	Basis	Tensor
First-order tensors $\in \mathbf{R}^N$	T^i, T_i	$\mathbf{g}_i$, $\mathbf{g}^i$	$\mathbf{T}^{(1)} = T^i\mathbf{g}_i, T_i\mathbf{g}^i$
Second-order tensors $\in \mathbf{R}^N \times \mathbf{R}^N$	$T^{ij}, T_{ij}, T^i_j, T_i^j$	$\mathbf{g}_i$, $\mathbf{g}^i$, $\mathbf{g}_j$, $\mathbf{g}^j$	$\mathbf{T}^{(2)} = T^{ij}\mathbf{g}_i\mathbf{g}_j, T_{ij}\mathbf{g}^i\mathbf{g}^j, T^i_j\mathbf{g}_i\mathbf{g}^j, T_i^j\mathbf{g}^i\mathbf{g}_j$
Third-order tensors $\in \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^N$	$T^{ijk}, T_{ijk}, T^{ik}_j, T^j_{ik}$	$\mathbf{g}_i$, $\mathbf{g}^i$, $\mathbf{g}_j$, $\mathbf{g}^j$, $\mathbf{g}_k$, $\mathbf{g}^k$	$\mathbf{T}^{(3)} = T^{ijk}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k, T_{ijk}\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k, T^{ik}_j\mathbf{g}_i\mathbf{g}_k\mathbf{g}^j, T^j_{ik}\mathbf{g}^i\mathbf{g}^k\mathbf{g}_j$
N-order tensors $\in \mathbf{R}^N \times \cdots \times \mathbf{R}^N$	$T^{ijk\ldots n}, T_{ijk\ldots n}, T^{ik\ldots n}_j, T^j_{ik\ldots n.}$	$\mathbf{g}_i$, $\mathbf{g}^i$, $\mathbf{g}_j$, $\mathbf{g}^j$, $\mathbf{g}_k$, $\mathbf{g}^k$, …, $\mathbf{g}_n$, $\mathbf{g}^n$	$\mathbf{T}^{(N)} = T^{ijk\ldots n}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_{k\ldots}\mathbf{g}_n, T^{ik\ldots n}_j\mathbf{g}_i\mathbf{g}_{k\ldots}\mathbf{g}_n\mathbf{g}^j$

$$\mathbf{T} = T^{ij}\mathbf{g}_i\,\mathbf{g}_j \equiv \sum_{j=1}^{N}\sum_{i=1}^{N} T^{ij}\mathbf{g}_i\,\mathbf{g}_j, \quad i;j\text{: dummy indices}$$

$$\mathbf{T}^j = T^{ij}\mathbf{g}_i \equiv \sum_{i=1}^{N} T^{ij}\mathbf{g}_i, \quad i\text{: dummy; } j\text{: free index}.$$

2.2 Definition of Tensors

The definition of tensors is based on multilinear algebra with a multilinear map. We consider the real vector spaces $\mathbf{U}_1,\ldots,\mathbf{U}_n$ and their respective dual vector spaces $\mathbf{V}_1,\ldots,\mathbf{V}_m$. Each of their vector spaces belongs to the finite N-dimensional space $\mathbf{R}^N$, the image vector space $\mathbf{W}$, to the real space $\mathbf{R}$. A **mixed tensor of type** $(\boldsymbol{m}, \boldsymbol{n})$ can be defined as a multilinear functional $\mathbf{T}$ that maps an $(m + n)$ tuple of vectors of the vector spaces $\mathbf{U}$ and $\mathbf{V}$ into $\mathbf{W}$ (Fecko 2011) (Fig. 2.1):

$$\begin{aligned} &\mathbf{T} : (\mathbf{U}_1 \times \cdots \times \mathbf{U}_n) \times (\mathbf{V}_1 \times \cdots \times \mathbf{V}_m) \to \mathbf{W} \\ &\underbrace{\mathbf{R}^N \times \cdots \times \mathbf{R}^N}_{n\,\text{copies}} \times \underbrace{\mathbf{R}^N \times \cdots \times \mathbf{R}^N}_{m\,\text{copies}} \to \mathbf{R} \\ &(\mathbf{u}_1,\ldots,\mathbf{u}_n;\mathbf{v}_1,\ldots,\mathbf{v}_m) \to \mathbf{T}(\mathbf{u}_1,\ldots,\mathbf{u}_n;\mathbf{v}_1,\ldots,\mathbf{v}_m) \in \mathbf{R}. \end{aligned} \tag{2.1}$$

Mapping the multilinear functional $\mathbf{T}$ of the tensor type (m, n) to the contravariant basis $\{\mathbf{g}^{im}\}$ of $\mathbf{U}$ and covariant basis $\{\mathbf{g}_{jn}\}$ of $\mathbf{V}$, one obtains its images in $\mathbf{W} \subset \mathbf{R}$. These images are called the components of the (m + n)-order mixed tensor $\mathbf{T}$ with respect to the relating bases:

$$T^{i1\ldots im}_{j1\ldots jn} \equiv \mathbf{T}(\mathbf{g}_{j1},\ldots,\mathbf{g}_{jn};\mathbf{g}^{i1},\ldots,\mathbf{g}^{im}) \in \mathbf{R} \tag{2.2}$$

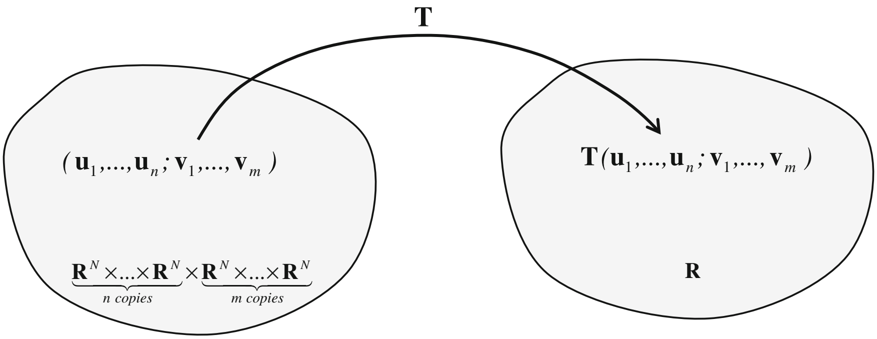

Fig. 2.1 Multilinear functional **T**

Thus, the $(m + n)$-order mixed tensor **T** can be expressed in the covariant and contravariant bases of the respective vector spaces **V** and **U**. In total, the $(m + n)$-order tensor **T** has $\mathrm{N}^{(m+n)}$ components, as shown in Eq. (2.3).

$$\begin{aligned} &\mathbf{T} = T^{i1\ldots im}_{j1\ldots jn}\, \mathbf{g}_{i1}\cdots\mathbf{g}_{im}\, \mathbf{g}^{j1}\cdots\mathbf{g}^{jn}; \\ &\mathbf{T} \in \underbrace{\mathbf{R}^N \times \cdots \times \mathbf{R}^N}_{n\,\text{copies}} \times \underbrace{\mathbf{R}^N \times \cdots \times \mathbf{R}^N}_{m\,\text{copies}} \end{aligned} \tag{2.3}$$

In the case of covariant and contravariant tensors **T**, the dual vector spaces **V** and real vector spaces **U** are omitted in Eq. (2.1), respectively.

- n-order covariant tensors:

$$\mathbf{T} = T_{j1\ldots jn}\, \mathbf{g}^{j1}\ldots\mathbf{g}^{jn} \in (\mathbf{U}_1 \times \cdots \times \mathbf{U}_n) \tag{2.4}$$

- m-order contravariant tensors:

$$\mathbf{T} = T^{il\ldots im}\, \mathbf{g}_{il}\cdots\mathbf{g}_{im} \in (\mathbf{V}_1 \times \cdots \times \mathbf{V}_m) \tag{2.5}$$

An Example of a Second-Order Covariant Tensor

An arbitrary vector **v** can be expressed in the covariant basis $\mathbf{g}_k$ in the N-dimensional vector space **V** as

$$\mathbf{v} = v^k \mathbf{g}_\mathrm{k} \quad \text{for} \quad k = 1, 2, \ldots, N \tag{2.6}$$

Applying the bilinear mapping **T** to the vector **v** and using the Kronecker delta, one obtains its mapping image **Tv**. Straightforwardly, this is a tensor of one lower order compared to the mapping tensor **T**.

$$
\begin{aligned}
\mathbf{Tv} &\equiv T_{ij}\mathbf{g}^i\mathbf{g}^j.(v^k\mathbf{g}_k) \\
&= T_{ij}v^k(\mathbf{g}^i.\mathbf{g}_k)\mathbf{g}^j \\
&= T_{ij}v^k\delta^i_k\mathbf{g}^j \ \text{ for } \ i = k \\
&= (T_{kj}v^k)\mathbf{g}^j \ \text{ for } \ j, k = 1, 2, \ldots, N \\
&\equiv T^*_j\mathbf{g}^j \ \text{ for } \ j = 1, 2, \ldots, N
\end{aligned}
\tag{2.7a}
$$

whereas the second-order covariant tensor $\mathbf{T}$ can be expressed as

$$
\begin{aligned}
&\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j \ \text{ for } \ i,j = 1,2,\ldots,N; \\
&\mathbf{T} \in \mathbf{R}^N \times \mathbf{R}^N.
\end{aligned}
$$

Note that in the case of a three-dimensional vector space $\mathbf{R}^3$ ($N = 3$), there are nine covariant components T_{ij}. The number of the tensor components can be calculated by N^{n} ($3^2 = 9$), in which n is the number of indices i and j (n = 2).

Obviously, that the mapping image $\mathbf{Tv}$ is also a tensor of one lower order compared to the tensor $\mathbf{T}$. The covariant tensor component T^*_j can be calculated by

$$
T^*_j \equiv T_{kj}v^k = \mathbf{g}_j.\mathbf{Tv}\,. \tag{2.7b}
$$

2.3 Tensor Algebra

2.3.1 General Bases in General Curvilinear Coordinates

The vector $\mathbf{r}$ can be written in Cartesian coordinates of Euclidean space $\mathbf{E}^3$, as displayed in Fig. 2.2.

$$
\mathbf{r} = x^i\mathbf{e}_i \tag{2.8}
$$

The differential d$\mathbf{r}$ results from Eq. (2.8) in

$$
\mathrm{d}\mathbf{r} = \mathbf{e}_i\mathrm{d}x^i = \frac{\partial\mathbf{r}}{\partial x^i}\mathrm{d}x^i \tag{2.9}
$$

Using the differentiation chain rule, the orthonormal bases $\mathbf{e}_i$ of the coordinates x^i are defined by

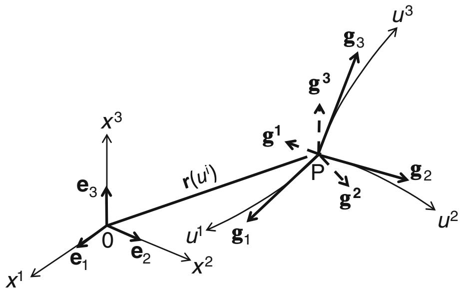

Fig. 2.2 Bases of general curvilinear coordinates in the space E^3

$$
\begin{aligned}
\mathbf{e}_{\mathrm{i}} &\equiv \frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial \mathbf{r}}{\partial u^{\mathrm{j}}}\frac{\partial u^j}{\partial x^i} \\
&\equiv \mathbf{g}_j \frac{\partial u^j}{\partial x^i} \quad \text{for } j = 1, 2, \ldots, N
\end{aligned}
\tag{2.10}
$$

Analogously, the bases of the curvilinear coordinates u^i can be calculated in the curvilinear coordinate system of $\mathbf{E}^N$

$$
\begin{aligned}
\mathbf{g}_{\mathrm{j}} &\equiv \frac{\partial \mathbf{r}}{\partial u^j} = \frac{\partial \mathbf{r}}{\partial x^k}\frac{\partial x^k}{\partial u^j} \\
&= \mathbf{e}_k \frac{\partial x^k}{\partial u^j} \quad \text{for } k = 1, 2, \ldots, N
\end{aligned}
\tag{2.11}
$$

The curvilinear coordinate u^i are functions of the coordinate x^i; the covariant bases in Eq. (2.11) can be calculated using the differentiation chain rule.

$$
\begin{aligned}
\mathbf{g}_j &= \frac{\partial \mathbf{r}}{\partial u^j} = \frac{\partial \mathbf{r}}{\partial x^i}\frac{\partial x^i}{\partial u^j} \\
&= \mathbf{e}_i \frac{\partial x^i}{\partial u^j} \\
&\equiv \mathbf{e}_i x^i_j \quad \text{for } i = 1, 2, \ldots, N
\end{aligned}
\tag{2.12}
$$

Thus, the curvilinear basis $\mathbf{g}_j$ can be written in a linear combination of the orthonormal basis $\mathbf{e}_i$ according to Eq. (2.12). The derivative x^i_j is called the shift tensor between the orthonormal and curvilinear coordinates.

Generally, the basis $\mathbf{g}_i$ of the curvilinear coordinate u^i can be rewritten in a linear combination of the basis $\mathbf{g}'_j$ of other curvilinear coordinate u'^j. The derivative u'^j_i is defined as the shift tensor between both curvilinear coordinates.

$$
\begin{aligned}
\mathbf{g}_i &= \frac{\partial \mathbf{r}}{\partial u'^j}\frac{\partial u'^j}{\partial u^i} \\
&= \mathbf{g}'_j \frac{\partial u'^j}{\partial u^i} \equiv \mathbf{g}'_j u'^j_i \quad \text{for } j = 1, 2, \ldots, N
\end{aligned}
\tag{2.13}
$$

In the curvilinear coordinate system (u^1, u^2, u^3) of Euclidean space $\mathbf{E}^3$, its basis is generally non-orthogonal and non-unitary (non-orthonormal basis); that is, the bases are not mutually perpendicular and their vector lengths are not equal to one (Simmonds 1982; Klingbeil 1966; Nayak 2012). In this case, the curvilinear coordinate system (u^1, u^2, u^3) has three covariant bases $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ and three contravariant bases $\mathbf{g}^1$, $\mathbf{g}^2$, and $\mathbf{g}^3$ at the origin P, as shown in Fig. 2.2. Generally, the origin P of the curvilinear coordinates could move everywhere in Euclidean space. Therefore, the bases of the curvilinear coordinates only depend on the respective origin P. For this reason, the curvilinear bases are not fixed in the whole curvilinear coordinates like in Cartesian coordinates.

The vector $\mathbf{r}$ of the point P(u^1, u^2, u^3) can be written in covariant and contravariant bases.

$$\begin{aligned} \mathbf{r} &= u^1\mathbf{g}_1 + u^2\mathbf{g}_2 + u^3\mathbf{g}_3 \\ &= u_1\mathbf{g}^1 + u_2\mathbf{g}^2 + u_3\mathbf{g}^3 \end{aligned} \tag{2.14}$$

where

u^1, u^2, u^3 are the vector contravariant components of the coordinates (u^1, u^2, u^3);
$\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ are the covariant bases of the coordinate system (u^1, u^2, u^3);
u_1, u_2, u_3 are the vector covariant components of the coordinates (u^1, u^2, u^3);
$\mathbf{g}^1$, $\mathbf{g}^2$, $\mathbf{g}^3$ are the contravariant bases of the coordinate system (u^1, u^2, u^3).

The covariant base $\mathbf{g}_i$ is defined by the tangential vector to the corresponding curvilinear coordinate u^i for $i = 1, 2, 3$. Both bases $\mathbf{g}_1$ and $\mathbf{g}_2$ generate a tangential surface to the curvilinear surface (u^1u^2) at the considered origin P. Note that the basis $\mathbf{g}_1$ is not perpendicular to the bases $\mathbf{g}_2$ and $\mathbf{g}_3$. However, the contravariant basis $\mathbf{g}^3$ is perpendicular to the tangential surface ($\mathbf{g}_1\mathbf{g}_2$) at the origin P. Generally, the contravariant basis ($\mathbf{g}^k$) results from the cross product of the other covariant bases ($\mathbf{g}_i \times \mathbf{g}_j$).

$$\alpha\,\mathbf{g}^k = \mathbf{g}_i \times \mathbf{g}_j \quad \text{for} \quad i, j, k = 1, 2, 3 \tag{2.15}$$

where α is a scalar factor.

Multiplying Eq. (2.15) by the covariant basis $\mathbf{g}_k$, the scalar factor α can be calculated as

$$\begin{aligned} \alpha(\mathbf{g}^k.\mathbf{g}_k) &= \alpha\,\delta_k^k = \alpha = (\mathbf{g}_i \times \mathbf{g}_j).\mathbf{g}_k \\ &\Rightarrow \alpha = (\mathbf{g}_i \times \mathbf{g}_j).\mathbf{g}_k \equiv [\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k] \end{aligned} \tag{2.16}$$

The scalar factor α equals the scalar triple product that is given in (Klingbeil 1966):

$$
\begin{aligned}
\alpha \equiv [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] &= (\mathbf{g}_i \times \mathbf{g}_j).\mathbf{g}_k = (\mathbf{g}_k \times \mathbf{g}_i).\mathbf{g}_j = (\mathbf{g}_j \times \mathbf{g}_k).\mathbf{g}_i \\
&= \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} g_{31} & g_{32} & g_{33} \\ g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \\ g_{11} & g_{12} & g_{13} \end{vmatrix}^{\frac{1}{2}} \\
&= \sqrt{\det(g_{ij})} \equiv \sqrt{g} = J
\end{aligned}
\tag{2.17}
$$

where J is defined as the Jacobian, as given in

$$
J \equiv \varepsilon_{ijk} \frac{\partial x^i}{\partial u^1} \frac{\partial x^j}{\partial u^2} \frac{\partial x^k}{\partial u^3} = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix}
\tag{2.18}
$$

Thus,

$$
\mathbf{g}^k = \frac{\varepsilon_{ijk}}{\sqrt{g}} (\mathbf{g}_i \times \mathbf{g}_j) = \frac{\varepsilon_{ijk}}{J} (\mathbf{g}_i \times \mathbf{g}_j)
\tag{2.19}
$$

where ε_{ijk} is the Levi-Civita permutation symbols in Eq. (A.5), cf. Appendix A.

According to Eq. (2.19), the contravariant basis $\mathbf{g}^k$ is perpendicular to both covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$. Additionally, the contravariant basis $\mathbf{g}^k$ is chosen such that the vector length of the contravariant basis equals the inversed vector length of its relating covariant basis.

Therefore,

$$
\mathbf{g}^k.\mathbf{g}_i = \frac{(\mathbf{g}_i \times \mathbf{g}_j).\mathbf{g}_i}{[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k]} = \delta_i^k
\tag{2.20}
$$

As a result, the relation between the contravariant and covariant bases is given in the general curvilinear coordinate system $(u^1, \ldots, u^N)$.

$$
\begin{cases} \mathbf{g}_i.\mathbf{g}^k = \mathbf{g}^k.\mathbf{g}_i = \delta_i^k & \text{for} \quad i, k = 1, 2, \ldots, N \\ \mathbf{g}_i.\mathbf{g}_k = \mathbf{g}_k.\mathbf{g}_i \neq \delta_i^k & \text{for} \quad i, k = 1, 2, \ldots, N \end{cases}
\tag{2.21}
$$

The basis $\mathbf{g}_i$ is called dual to the basis $\mathbf{g}^j$ (Itskov 2010) if

$$
\mathbf{g}_i.\mathbf{g}^j = \delta_i^j \quad \text{for} \quad i, j = 1, 2, \ldots, N
\tag{2.22}
$$

where δ_i^j is the Kronecker delta.

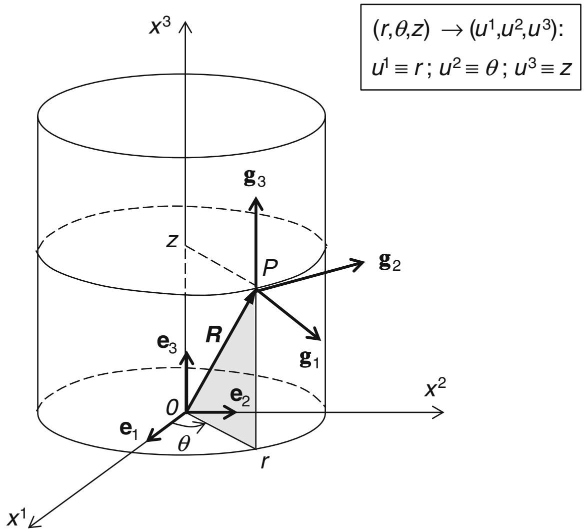

Fig. 2.3 Covariant bases of orthogonal cylindrical coordinates

Let $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_N\}$ be a covariant basis of the curvilinear coordinates $\{u^i\}$, the contravariant basis $\{\mathbf{g}^1, \mathbf{g}^2, \ldots, \mathbf{g}^N\}$, the dual basis to the covariant basis, can be written in the matrix formulation.

$$\mathbf{G} = [\mathbf{g}_1 \quad \mathbf{g}_2 \quad . \quad \mathbf{g}_i \quad \mathbf{g}_N]; \mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ . \\ \mathbf{g}^j \\ \mathbf{g}^N \end{bmatrix} \Rightarrow \mathbf{G}^{-1}\mathbf{G} = \mathbf{I} \tag{2.23}$$

where $\mathbf{g}^j$ is the jth row vector of $\mathbf{G}^{-1}$; $\mathbf{g}_i$ is the ith column vector of $\mathbf{G}$.

The covariant and contravariant bases (dual bases) of the orthogonal cylindrical and spherical coordinates are computed in the following section.

(a) Orthogonal Cylindrical Coordinates

The cylindrical coordinates (r, θ, z) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 2.3 shows a point P in the cylindrical coordinates (r, θ, z) embedded in the orthonormal Cartesian coordinates (x^1, x^2, x^3). However, the cylindrical coordinates change as the point P varies.

The vector **OP** can be written in Cartesian coordinates (x^1, x^2, x^3):

$$\begin{aligned} \mathbf{R} &= (r\cos\theta)\mathbf{e}_1 + (r\sin\theta)\mathbf{e}_2 + z\,\mathbf{e}_3 \\ &\equiv x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \end{aligned} \tag{2.24}$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the orthonormal bases of Cartesian coordinates;
θ is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 are used for r, θ, and z, respectively. Therefore, the coordinates of $P(u^1, u^2, u^3)$ are given in Cartesian coordinates:

$$P(u^1, u^2, u^3) = \begin{Bmatrix} x^1 = r\cos\theta \equiv u^1 \cos u^2 \\ x^2 = r\sin\theta \equiv u^1 \sin u^2 \\ x^3 = z \equiv u^3 \end{Bmatrix} \tag{2.25}$$

The covariant bases of the curvilinear coordinates are computed from

$$\mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{R}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} = \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \quad \text{for} \quad j = 1, 2, 3 \tag{2.26}$$

The covariant basis matrix **G** yields from Eq. (2.26):

$$\begin{aligned} \mathbf{G} &= [\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3] \\ &= \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{2.27}$$

The determinant of **G** is called the Jacobian J.

$$|\mathbf{G}| \equiv J = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \tag{2.28}$$

The relation between the covariant and contravariant bases yields from Eq. (2.22):

$$\mathbf{g}^i . \mathbf{g}_j = \delta^i_j \,(\text{Kronecker delta}) \tag{2.29}$$

Thus, the contravariant basis matrix $\mathbf{G}^{-1}$ results from the inversion of the covariant basis matrix **G**, as given in Eq. (2.27).

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & r \end{pmatrix} \tag{2.30}$$

The calculation of the determinant and inversion matrix of $\mathbf{G}$ will be discussed in the following section.

According to Eq. (2.27), the covariant bases can be denoted as

$$\begin{cases} \mathbf{g}_1 = (\cos\theta)\,\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2 + 0.\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = 1 \\ \mathbf{g}_2 = -(r\sin\theta)\,\mathbf{e}_1 + (r\cos\theta)\,\mathbf{e}_2 + 0.\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = r \\ \mathbf{g}_3 = 0.\,\mathbf{e}_1 + 0..\mathbf{e}_2 + 1.\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = 1 \end{cases} \tag{2.31}$$

The contravariant bases result from Eq. (2.30).

$$\begin{cases} \mathbf{g}^1 = (\cos\theta)\,\mathbf{e}_1 + (\sin\theta)\,\mathbf{e}_2 + 0.\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^1\right| = 1 \\ \mathbf{g}^2 = -\left(\dfrac{\sin\theta}{r}\right)\mathbf{e}_1 + \left(\dfrac{\cos\theta}{r}\right)\mathbf{e}_2 + 0.\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^2\right| = \dfrac{1}{r} \\ \mathbf{g}^3 = 0\,.\mathbf{e}_1 + 0.\,\mathbf{e}_2 + 1.\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^3\right| = 1 \end{cases} \tag{2.32}$$

Not only the covariant bases but also the contravariant bases of the cylindrical coordinates are orthogonal due to

$$\begin{aligned} &\mathbf{g}_i.\mathbf{g}^j = \mathbf{g}^j.\mathbf{g}_i = \delta_i^j \\ &\mathbf{g}_i.\mathbf{g}_j = 0 \;\text{ for }\; i \neq j; \\ &\mathbf{g}^i.\mathbf{g}^j = 0 \;\text{ for }\; i \neq j. \end{aligned}$$

(b) **Orthogonal Spherical Coordinates**

The spherical coordinates (ρ, ϕ, θ) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary.

The spherical coordinates (ρ, ϕ, θ) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 2.4 shows a point P in the spherical coordinates (r, θ, z) embedded in the orthonormal Cartesian coordinates (x^1, x^2, x^3). However, the spherical coordinates change as the point P varies.

The vector **OP** can be written in Cartesian coordinates (x^1, x^2, x^3):

$$\begin{aligned} \mathbf{R} = &\ (\rho\sin\phi\cos\theta)\,\mathbf{e}_1 + (\rho\sin\phi\sin\theta)\,\mathbf{e}_2 + \rho\cos\phi\,\mathbf{e}_3 \\ &\equiv x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \end{aligned} \tag{2.33}$$

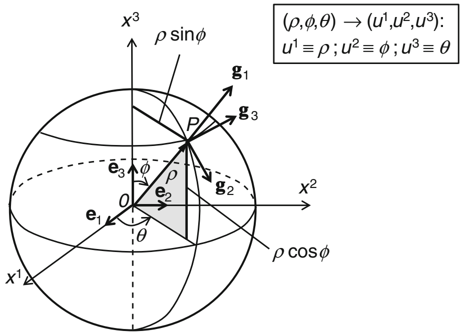

Fig. 2.4 Covariant bases of orthogonal spherical coordinates

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$	are the orthonormal bases of Cartesian coordinates;
ϕ	is the equatorial angle;
θ	is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 are used for ρ, ϕ, and θ, respectively. Therefore, the coordinates of $P(u^1, u^2, u^3)$ are given in Cartesian coordinates:

$$P(u^1,u^2,u^3) = \left\{ \begin{aligned} x^1 &= \rho \sin\phi \cos\theta \equiv u^1 \sin u^2 \cos u^3 \\ x^2 &= \rho \sin\phi \sin\theta \equiv u^1 \sin u^2 \cos u^3 \\ x^3 &= \rho \cos\phi \equiv u^1 \cos u^2 \end{aligned} \right\} \tag{2.34}$$

The covariant bases of the curvilinear coordinates are computed from

$$\mathbf{g}_i = \frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{R}}{\partial x^j} \cdot \frac{\partial x^j}{\partial u^i} = \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \quad \text{for} \quad j = 1,2,3 \tag{2.35}$$

Thus, the covariant basis matrix **G** can be calculated from Eq. (2.35).

$$\mathbf{G} = [\,\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3\,] = \begin{pmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix} \tag{2.36}$$

The determinant of the covariant basis matrix **G** is called the Jacobian J.

$$\begin{aligned}|\mathbf{G}| \equiv J &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\ &= \rho^2\sin\phi \end{aligned} \tag{2.37}$$

Similarly, the contravariant basis matrix $\mathbf{G}^{-1}$ is the inversion of the covariant basis matrix **G**.

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\phi\cos\theta & \rho\sin\phi\sin\theta & \rho\cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\left(\dfrac{\sin\theta}{\sin\phi}\right) & \left(\dfrac{\cos\theta}{\sin\phi}\right) & 0 \end{pmatrix} \tag{2.38}$$

The matrix product $\mathbf{G}^{-1}\cdot\mathbf{G}$ must be an identity matrix according to Eq. (2.23).

$$\mathbf{G}^{-1}\mathbf{G} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\phi\cos\theta & \rho\sin\phi\sin\theta & \rho\cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\left(\dfrac{\sin\theta}{\sin\phi}\right) & \left(\dfrac{\cos\theta}{\sin\phi}\right) & 0 \end{pmatrix}$$
$$\cdot\begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \mathbf{I} \tag{2.39}$$

According to Eq. (2.36), the covariant bases can be written as

$$\begin{aligned} \mathbf{g}_1 &= (\sin\phi\cos\theta)\,\mathbf{e}_1 + (\sin\phi\sin\theta)\,\mathbf{e}_2 + \cos\phi\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = 1 \\ \mathbf{g}_2 &= (\rho\cos\phi\cos\theta)\,\mathbf{e}_1 + (\rho\cos\phi\sin\theta)\,\mathbf{e}_2 - (\rho\sin\phi)\,\mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = \rho \\ \mathbf{g}_3 &= (-\rho\sin\phi\sin\theta)\,\mathbf{e}_1 + (\rho\sin\phi\cos\theta)\,\mathbf{e}_2 + 0.\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = \rho\sin\phi \end{aligned} \tag{2.40}$$

The contravariant bases result from Eq. (2.38).

$$\begin{aligned} \mathbf{g}^1 &= (\sin\phi\cos\theta)\,\mathbf{e}_1 + (\sin\phi\sin\theta)\,\mathbf{e}_2 + \cos\phi\,\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^1\right| = 1 \\ \mathbf{g}^2 &= \left(\frac{1}{\rho}\cos\phi\cos\theta\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\cos\phi\sin\theta\right)\mathbf{e}_2 - \left(\frac{1}{\rho}\sin\phi\right)\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^2\right| = \frac{1}{\rho} \\ \mathbf{g}^3 &= -\left(\frac{1}{\rho}\frac{\sin\theta}{\sin\phi}\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\frac{\cos\theta}{\sin\phi}\right)\mathbf{e}_2 + 0.\mathbf{e}_3 \Rightarrow \left|\mathbf{g}^3\right| = \frac{1}{\rho\sin\phi} \end{aligned} \tag{2.41}$$

Not only the covariant bases but also the contravariant bases of the spherical coordinates are orthogonal due to

$$\begin{aligned}\mathbf{g}_i.\mathbf{g}^j &= \mathbf{g}^j.\mathbf{g}_i = \delta_i^j\\ \mathbf{g}_i.\mathbf{g}_j &= 0 \ \text{ for } \ i \neq j;\\ \mathbf{g}^i.\mathbf{g}^j &= 0 \ \text{ for } \ i \neq j.\end{aligned}$$

2.3.2 Metric Coefficients in General Curvilinear Coordinates

The covariant basis vectors $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ to the general curvilinear coordinates (u^1, u^2, u^3) at the point P can be calculated from the orthonormal bases ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) in Cartesian coordinates $x^j = x^j(u^i)$, as shown in Fig. 2.2.

$$\begin{aligned}\mathbf{g}_i &\equiv \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial \mathbf{r}}{\partial x^k}\cdot\frac{\partial x^k}{\partial u^i}\\ &= \mathbf{e}_k\frac{\partial x^k}{\partial u^i} \quad \text{for} \quad k = 1,2,3\end{aligned} \tag{2.42}$$

The covariant metric coefficients g_{ij} are defined as

$$\begin{aligned}g_{ij} &\equiv \mathbf{g}_i.\mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^i}\frac{\partial \mathbf{r}}{\partial u^j} = \mathbf{g}_j.\mathbf{g}_i \equiv g_{ji}\\ &= \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\mathbf{e}_k\mathbf{e}_l = \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\delta_{kl}\\ &= \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}\end{aligned} \tag{2.43}$$

Similarly, the contravariant metric coefficients g^{ij} can be denoted as

$$g^{ij} \equiv \mathbf{g}^i.\mathbf{g}^j = \mathbf{g}^j.\mathbf{g}^i = g^{ji} \tag{2.44}$$

Furthermore, the contravariant basis can be rewritten as a linear combination of the covariant bases.

$$\mathbf{g}^i = A^{ij}\mathbf{g}_j \tag{2.45}$$

According to Eq. (2.44) and using Eqs. (2.21) and (2.45), the contravariant metric coefficients can be expressed as

$$\begin{aligned} g^{ik} &\equiv \mathbf{g}^i . \mathbf{g}^k = A^{ij} \mathbf{g}_{\mathrm{j}} . \mathbf{g}^k \\ &= A^{ij} \delta_j^k = A^{ik} \end{aligned} \tag{2.46}$$

Thus,

$$\mathbf{g}^i = g^{ij} \mathbf{g}_j \;\text{ for }\; j = 1, 2, 3 \tag{2.47}$$

Analogously, one obtains the covariant basis

$$\mathbf{g}_k = g_{kl} \mathbf{g}^l \;\text{ for }\; l = 1, 2, 3 \tag{2.48}$$

The mixed metric coefficients can be defined by

$$\begin{aligned} g_k^i &\equiv \mathbf{g}^i . \mathbf{g}_k = \left(g^{ij} \mathbf{g}_j\right) . \mathbf{g}_k \\ &= \left(g^{ij} \mathbf{g}_j\right) . \left(g_{kl} \mathbf{g}^l\right) = g^{ij} g_{kl} \left(\mathbf{g}_j . \mathbf{g}^l\right) \\ &= g^{ij} g_{\mathrm{kl}} \delta_j^l = g^{ij} g_{kj} \\ &= \delta_k^i \end{aligned} \tag{2.49a}$$

Thus,

$$g^{ij} g_{kj} = g_{kj} g^{ij} = \delta_k^i \tag{2.49b}$$

Therefore, the contravariant metric tensor is the inverse of the covariant metric tensor.

$$g^{ij} g_{kj} = g_{kj} g^{ij} = \delta_k^i \;\Leftrightarrow \mathbf{M}^{-1} \mathbf{M} = \mathbf{M} \mathbf{M}^{-1} = \mathbf{I} \tag{2.50}$$

where $\mathbf{M}^{-1}$ and $\mathbf{M}$ are the contravariant and covariant metric tensors.

Thus,

$$\begin{aligned} \mathbf{M}^{-1} \mathbf{M} &= \begin{bmatrix} g^{11} & g^{12} & . & g^{1N} \\ g^{21} & g^{22} & . & g^{2N} \\ . & . & g^{ij} & . \\ g^{N1} & g^{N2} & . & g^{NN} \end{bmatrix} . \begin{bmatrix} g_{11} & g_{12} & . & g_{1N} \\ g_{21} & g_{22} & . & g_{2N} \\ . & . & g_{ij} & . \\ g_{N1} & g_{N2} & . & g_{NN} \end{bmatrix} \\ &= \left(g^{ij} g_{kj}\right) = \left(\delta_k^i\right) = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ 0 & . & 1 & 0 \\ 0 & 0 & . & 1 \end{bmatrix} = \mathbf{I} \end{aligned} \tag{2.51}$$

According to Eqs. (2.42) and (2.49a), the contravariant bases of the curvilinear coordinates can be derived as

$$
\begin{aligned}
\delta_i^k &\equiv \mathbf{g}^k.\mathbf{g}_i = \mathbf{g}^k.\frac{\partial x^j}{\partial u^i}\mathbf{e}_j \Rightarrow \\
\delta_i^k.\mathbf{e}_j &= \left(\mathbf{g}^k.\frac{\partial x^j}{\partial u^i}\mathbf{e}_j\right).\mathbf{e}_j = \mathbf{g}^k.\frac{\partial x^j}{\partial u^i} \Rightarrow \\
\mathbf{g}^k &= \delta_i^k \frac{\partial u^i}{\partial x^j}\mathbf{e}_j = \frac{\partial u^k}{\partial x^j}\mathbf{e}_j
\end{aligned}
\tag{2.52}
$$

Thus,

$$
\mathbf{g}^j = \frac{\partial u^j}{\partial x^i}\mathbf{e}_i \ \text{ for } \ i = 1, 2, 3 \tag{2.53}
$$

Generally, the covariant and contravariant metric coefficients of the general curvilinear coordinates have the following properties:

$$
\begin{cases}
g_{ji} = g_{ij} = \mathbf{g}_i.\mathbf{g}_j \neq \delta_i^j (\text{cov. metric coefficient}) \\
g^{ji} = g^{ij} = \mathbf{g}^i.\mathbf{g}^j \neq \delta_i^j (\text{contrav. metric coefficient}) \\
g_i^j = \mathbf{g}_i.\mathbf{g}^j = \delta_i^j (\text{mixed metric coefficient})
\end{cases}
\tag{2.54}
$$

Using Eqs. (2.42) and (2.53), one obtains the Kronecker delta

$$
\begin{aligned}
\delta_i^j &= \mathbf{g}_i.\mathbf{g}^j = \frac{\partial x^k}{\partial u^i}\frac{\partial u^j}{\partial x^l}\mathbf{e}_k.\mathbf{e}_l = \frac{\partial x^k}{\partial u^i}\frac{\partial u^j}{\partial x^l}\delta_k^l \\
&= \frac{\partial x^k}{\partial u^i}\frac{\partial u^j}{\partial x^k} = \frac{\partial u^j}{\partial u^i}.
\end{aligned}
\tag{2.55a}
$$

The Kronecker delta is defined by

$$
\delta_i^j \equiv \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases} \tag{2.55b}
$$

As an example, the covariant metric tensor **M** in the *cylindrical coordinates* results from Eq. (2.31).

$$
\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.56a}
$$

The contravariant metric coefficients in the contravariant metric tensor $\mathbf{M}^{-1}$ are calculated from inverting the covariant metric tensor **M**.

$$\mathbf{M}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.56b}$$

Analogously, the covariant metric tensor **M** in the *spherical coordinates* results from Eq. (2.40).

$$\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & (\rho \sin \phi)^2 \end{bmatrix} \tag{2.57a}$$

The contravariant metric coefficients in the contravariant metric tensor $\mathbf{M}^{-1}$ are calculated from inverting the covariant metric tensor **M**.

$$\mathbf{M}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & (\rho \sin \phi)^{-2} \end{bmatrix} \tag{2.57b}$$

2.3.3 Tensors of Second Order and Higher Orders

Mapping an arbitrary vector $\mathbf{x} \in \mathbf{R}^N$ by a linear functional **T**, one obtains its image vector $\mathbf{y} = \mathbf{Tx}$ (Simmonds 1982; Klingbeil 1966).

$$\begin{aligned} &\mathbf{T} : \mathbf{R}^N \rightarrow \mathbf{R}^N \\ &\mathbf{T} : \mathbf{x} \rightarrow \mathbf{y} = \mathbf{Tx} \equiv \mathbf{T}.\left(x^j \mathbf{g}_j\right) = x^j \mathbf{T}.\mathbf{g}_j \\ &\qquad = x^j T_{ik} \mathbf{g}^i \left(\mathbf{g}^k . \mathbf{g}_j\right) = x^j T_{ij} \delta_j^k \mathbf{g}^{\mathrm{i}} = T_{ik} x^k \mathbf{g}^i \end{aligned} \tag{2.58}$$

where **T** is a second-order tensor $\in \mathbf{R}^N \times \mathbf{R}^N$.

It is obvious that the image vector $\mathbf{y} = \mathbf{Tx}$ is a tensor of one lower order compared to the tensor **T** that can be considered as a linear operator.

The second-order tensor **T** can be generated from the tensor product (dyadic product) of two vectors **u** and **v**, as denoted in Eq. (2.60).

Let **u** and **v** be two arbitrary vectors, they can be written in the covariant and contravariant bases as

$$\begin{cases} \mathbf{u} = u^i \mathbf{g}_i = u_i \mathbf{g}^i \in \mathbf{R}^N; \\ \mathbf{v} = v^j \mathbf{g}_j = v_j \mathbf{g}^j \in \mathbf{R}^N \end{cases} \tag{2.59}$$

The tensor product of two vectors $\mathbf{u} \cdot \mathbf{v}$ results in the second-order tensor **T**.

$$
\begin{aligned}
&\mathbf{T}:\mathbf{u},\mathbf{v}\in\mathbf{R}^N \rightarrow \mathbf{T}\equiv\mathbf{u}\otimes\mathbf{v}\in\mathbf{R}^N\times\mathbf{R}^N\\
&\Rightarrow \mathbf{T}=u^iv^j\mathbf{g}_i\mathbf{g}_j=u^iv^j\mathbf{g}_i\mathbf{g}_j\equiv T^{ij}\mathbf{g}_i\mathbf{g}_j\\
&\Rightarrow \mathbf{T}=u_iv_j\mathbf{g}^i\mathbf{g}^j=u_iv_j\mathbf{g}^i\mathbf{g}^j\equiv T_{ij}\mathbf{g}^i\mathbf{g}^j
\end{aligned}
\tag{2.60}
$$

Note that the terms $\mathbf{g}_i\mathbf{g}_j$ and $\mathbf{g}^i\mathbf{g}^j$ are called the covariant and contravariant basis tensors, respectively. Hence, they are not the same notations as the covariant and contravariant metric coefficients g_{ij} and g^{ij}, respectively.

$$
\begin{aligned}
\mathbf{g}_i\mathbf{g}_j &\neq \mathbf{g}_i.\mathbf{g}_j \equiv g_{ij}\\
\mathbf{g}^i\mathbf{g}^j &\neq \mathbf{g}^i.\mathbf{g}^j \equiv g^{ij}
\end{aligned}
$$

Similarly, one obtains the properties of the mixed basis tensors:

$$
\begin{aligned}
\mathbf{g}^i\mathbf{g}_j &\neq \mathbf{g}^i.\mathbf{g}_j \equiv g^i_j = \delta^i_j\\
\mathbf{g}_i\mathbf{g}^j &\neq \mathbf{g}_i.\mathbf{g}^j \equiv g^j_i = \delta^j_i
\end{aligned}
$$

Each of the covariant and contravariant tensor components T_{ij} and T^{ij} contains nine independent elements ($N^2 = 9$) in a nine-dimensional tensor space $\mathbf{R}^3 \times \mathbf{R}^3$ in a three-dimensional space ($N = 3$).

$$
\begin{aligned}
T^{ij} &= u^iv^j;\\
T_{ij} &= u_iv_j
\end{aligned}
\tag{2.61}
$$

The basis $\mathbf{g}_j$ of the general curvilinear coordinates is mapped by the linear functional $\mathbf{T}$ in Eq. (2.58) into the image vector $\mathbf{T}_j$ that can be written according to Eq. (2.2) as

$$
\mathbf{T}_j \equiv \mathbf{T}.\mathbf{g}_j \tag{2.62}
$$

Each vector $\mathbf{T}_j$ can be expressed in a linear combination of the contravariant basis $\mathbf{g}^i$ as

$$
\mathbf{T}_j = T_{ij}\mathbf{g}^i \tag{2.63}
$$

where T_{ij} is the covariant tensor component of the second-order tensor $\mathbf{T}$.

Multiplying Eq. (2.63) by the covariant basis $\mathbf{g}_i$ and using Eq. (2.62), the covariant tensor component T_{ij} results in

$$
\begin{aligned}
&\left(T_{ij}\mathbf{g}^i\right).\mathbf{g}_i = \left(\mathbf{T}.\mathbf{g}_j\right).\mathbf{g}_i = \mathbf{g}_i.\mathbf{T}.\mathbf{g}_j\\
&\qquad\Rightarrow T_{ij}\delta^i_i = T_{ij} = \mathbf{g}_i.\mathbf{T}.\mathbf{g}_j
\end{aligned}
\tag{2.64}
$$

Equation (2.64) can be written in the contravariant bases $\mathbf{g}^i$ and $\mathbf{g}^j$ as follows:

$$\begin{aligned} T_{ij}\left(\mathbf{g}^i\mathbf{g}^j\right) &= \mathbf{g}_i.\mathbf{T}.\mathbf{g}_j\left(\mathbf{g}^i\mathbf{g}^j\right) \\ &= \left(\mathbf{g}_i.\mathbf{g}^i\right)\mathbf{T}\left(\mathbf{g}^j.\mathbf{g}_j\right) = \delta_i^i\mathbf{T}\delta_j^j = \mathbf{T} \\ &\Rightarrow \mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j \end{aligned} \tag{2.65}$$

Similarly, the vector $\mathbf{T}^j$ is formulated in a linear combination of the covariant basis $\mathbf{g}_i$.

$$\mathbf{T}^j = \mathbf{T}.\mathbf{g}^j = T^{ij}\mathbf{g}_i \tag{2.66}$$

in which T^{ij} is the contravariant component of the second-order tensor $\mathbf{T}$.

Multiplying Eq. (2.66) by the contravariant basis $\mathbf{g}^i$, the contravariant tensor component T^{ij} can be computed as

$$\begin{aligned} \left(T^{ij}\mathbf{g}_i\right).\mathbf{g}^i &= \left(\mathbf{T}.\mathbf{g}^j\right).\mathbf{g}^i = \mathbf{g}^i.\mathbf{T}.\mathbf{g}^j \\ &\Rightarrow T^{ij}\delta_i^i = T^{ij} = \mathbf{g}^i.\mathbf{T}.\mathbf{g}^j \end{aligned} \tag{2.67}$$

Similarly, Eq. (2.67) can be written in the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$

$$\begin{aligned} T^{ij}\left(\mathbf{g}_i\mathbf{g}_j\right) &= \mathbf{g}^i.\mathbf{T}.\mathbf{g}^j\left(\mathbf{g}_i\mathbf{g}_j\right) \\ &= \left(\mathbf{g}^i.\mathbf{g}_i\right)\mathbf{T}\left(\mathbf{g}_j.\mathbf{g}^j\right) = \delta_i^i\mathbf{T}\delta_j^j = \mathbf{T} \\ &\Rightarrow \mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j \end{aligned} \tag{2.68}$$

Alternatively, the vector component can be rewritten in a linear combination of the mixed tensor component.

$$\mathbf{T}^j = \mathbf{T}.\mathbf{g}^j = T_{i.}^{\,j}\mathbf{g}^i \tag{2.69}$$

Multiplying Eq. (2.69) by the covariant basis $\mathbf{g}_i$, the mixed tensor component $T_{i.}^{\,j}$ can be calculated as

$$\begin{aligned} \left(T_{i.}^{\,j}\mathbf{g}^i\right).\mathbf{g}_i &= \left(\mathbf{T}.\mathbf{g}^j\right).\mathbf{g}_i = \mathbf{g}_i.\mathbf{T}.\mathbf{g}^j \\ &\Rightarrow T_{i.}^{\,j}\delta_i^i = T_{i.}^{\,j} = \mathbf{g}_i.\mathbf{T}.\mathbf{g}^j \end{aligned} \tag{2.70}$$

Note that in Eq. (2.70), the dot *after* the lower index indicates the position of the basis of the upper index locating *after* the tensor $\mathbf{T}$. In this case, the tensor $\mathbf{T}$ is located between the lower basis $\mathbf{g}_i$ and upper basis $\mathbf{g}^j$ (Itskov 2010; Nayak 2012; Oeijord 2005).

Equation (2.70) can be written in the covariant and contravariant bases $\mathbf{g}_j$ and $\mathbf{g}^i$ as follows:

$$
\begin{aligned}
T_{i.}^{\ j}\left(\mathbf{g}^i\mathbf{g}_j\right) &= \mathbf{g}_i.\mathbf{T}.\mathbf{g}^j\left(\mathbf{g}^i\mathbf{g}_j\right) \\
&= \left(\mathbf{g}_i.\mathbf{g}^i\right)\mathbf{T}\left(\mathbf{g}_j.\mathbf{g}^j\right) = \delta_i^i\mathbf{T}\delta_j^j = \mathbf{T} \\
&\Rightarrow \mathbf{T} = T_{i.}^{\ j}\mathbf{g}^i\mathbf{g}_j
\end{aligned} \tag{2.71}
$$

Analogously, one obtains the mixed tensor component $T^{i}_{.j}$.

$$
\begin{aligned}
\left(T^{i}_{.j}\mathbf{g}^j\right).\mathbf{g}_j &= \left(\mathbf{T}.\mathbf{g}^i\right).\mathbf{g}_j = \mathbf{T}.\left(\mathbf{g}^i.\mathbf{g}_j\right) \\
&= \mathbf{T}.\left(\mathbf{g}_j.\mathbf{g}^i\right) = \mathbf{g}^i.\mathbf{T}.\mathbf{g}_j \\
&\Rightarrow T^{i}_{.j}\delta_j^j = T^{i}_{.j} = \mathbf{g}^i.\mathbf{T}.\mathbf{g}_j
\end{aligned} \tag{2.72}
$$

Note that in Eq. (2.72), the dot *before* the lower index indicates the position of the basis of the upper index locating *in front of* the tensor **T**. In this case, the tensor **T** is located between the upper basis $\mathbf{g}^i$ and lower basis $\mathbf{g}_j$ (Itskov 2010; Nayak 2012; Oeijord 2005).

Equation (2.72) can be written in the covariant and contravariant bases $\mathbf{g}_i$ and $\mathbf{g}^j$ as follows:

$$
\begin{aligned}
T^{i}_{.j}\left(\mathbf{g}_i\mathbf{g}^j\right) &= \mathbf{g}^i.\mathbf{T}.\mathbf{g}_j\left(\mathbf{g}_i\mathbf{g}^j\right) \\
&= \left(\mathbf{g}^i.\mathbf{g}_i\right)\mathbf{T}\left(\mathbf{g}^j.\mathbf{g}_j\right) = \delta_i^i\mathbf{T}\delta_j^j = \mathbf{T} \\
&\Rightarrow \mathbf{T} = T^{i}_{.j}\mathbf{g}_i\mathbf{g}^j
\end{aligned} \tag{2.73}
$$

In a nutshell, the second-order tensor can be written in different expressions according to the covariant, contravariant, and mixed components.

$$
\mathbf{T}^{(2)} = \begin{cases}
T_{ij}\mathbf{g}^i\mathbf{g}^j;\ T_{ij} = \mathbf{g}_i.\mathbf{T}.\mathbf{g}_j \\
T^{ij}\mathbf{g}_i\mathbf{g}_j;\ T^{ij} = \mathbf{g}^i.\mathbf{T}.\mathbf{g}^j \\
T_{i.}^{\ j}\mathbf{g}^i\mathbf{g}_j;\ T_{i.}^{\ j} = \mathbf{g}_i.\mathbf{T}.\mathbf{g}^j \\
T^{i}_{.j}\mathbf{g}_i\mathbf{g}^j;\ T^{i}_{.j} = \mathbf{g}^i.\mathbf{T}.\mathbf{g}_j
\end{cases} \tag{2.74}
$$

Note that if the second-order tensor **T** is symmetric, then

$$
T_{ij} = T_{ji};\ T^{ij} = T^{ji};\ T_{i.}^{\ j} = T^{j}_{.i};\ T^{i}_{.j} = T_{j.}^{\ i} \tag{2.75}
$$

Compared to the second-order tensors, the first-order tensor $\mathbf{T}^{(1)}$ has only one dummy index, as shown in

$$
\mathbf{T}^{(1)} = \begin{cases}
T_i\mathbf{g}^i;\ T_{\mathrm{i}} = \mathbf{T}.\mathbf{g}_i \\
T^i\mathbf{g}_i;\ T^i = \mathbf{T}.\mathbf{g}^i
\end{cases} \tag{2.76}
$$

An N-order tensor $\mathbf{T}^{(N)}$ is the tensor product of the N covariant, contravariant, and mixed bases of the coordinates:

$$\mathbf{T}^{(N)} = \begin{cases} T^{ij\ldots n}\mathbf{g}_i\mathbf{g}_j\ldots\mathbf{g}_n \\ T_{ij\ldots n}\mathbf{g}^i\mathbf{g}^j\ldots\mathbf{g}^n \\ T^{ij..}_{l\ldots n}\mathbf{g}_i\mathbf{g}_j\ldots\mathbf{g}^l\ldots\mathbf{g}^n \end{cases} \tag{2.77}$$

The N-order tensors contain the 2^N expressions in total. Two of them in respect of the covariant and contravariant tensor components and $(2^N - 2)$ expressions in respect of the mixed tensor components (Klingbeil 1966). In the case of a second-order tensor $\mathbf{T}^{(2)}$ for $N = 2$, there are four expressions: two with the covariant and contravariant tensor components and two with the mixed tensor components, as displayed in Eq. (2.74).

2.3.4 Tensor and Cross Products of Two Vectors in General Bases

(a) Tensor product

Let $\mathbf{u}$ and $\mathbf{v}$ be two arbitrary vectors in the finite N-dimensional vector space $\mathbf{R}^N$, they can be written in the covariant and contravariant bases as

$$\begin{cases} \mathbf{u} = u^i\mathbf{g}_i = u_i\mathbf{g}^i \in \mathbf{R}^N; \\ \mathbf{v} = v^j\mathbf{g}_j = v_j\mathbf{g}^j \in \mathbf{R}^N \end{cases} \tag{2.78}$$

The tensor product $\mathbf{T}$ of two vectors generates a second-order tensor that can be defined by the linear functional $\mathbf{T}$.

- In the covariant bases:

$$\begin{aligned} \mathbf{T} : \mathbf{u}, \mathbf{v} \in \mathbf{R}^N \to \mathbf{T} \equiv \mathbf{u} \otimes \mathbf{v} = u^i v^j \mathbf{g}_i\mathbf{g}_j = u_i v_j \mathbf{g}^i\mathbf{g}^j \in \mathbf{R}^N \times \mathbf{R}^N \\ \mathbf{T} = u^i v^j \mathbf{g}_i\mathbf{g}_j \equiv T^{ij}\mathbf{g}_i\mathbf{g}_j \\ = u_i v_j \mathbf{g}^i\mathbf{g}^j \equiv T_{ij}\mathbf{g}^i\mathbf{g}^j \end{aligned} \tag{2.79}$$

where
T^{ij} is the contravariant component of the second-order tensor $\mathbf{T}$;
T_{ij} is the covariant component of the second-order tensor $\mathbf{T}$;

- In the covariant and contravariant bases:

$$\begin{aligned} \mathbf{T} : \mathbf{u}, \mathbf{v} \in \mathbf{R}^N \to \mathbf{T} \equiv \mathbf{u} \otimes \mathbf{v} = u^i v_j \mathbf{g}_i\mathbf{g}^j = u_i v^j \mathbf{g}^i\mathbf{g}_j \in \mathbf{R}^N \times \mathbf{R}^N \\ \mathbf{T} = u^i v_j \mathbf{g}_i\mathbf{g}^j \equiv T^i_j\mathbf{g}_i\mathbf{g}^j \\ = u_i v^j \mathbf{g}^i\mathbf{g}_j \equiv T_i^j\mathbf{g}^i\mathbf{g}_j \end{aligned} \tag{2.80}$$

where T_j^i and T_i^j are the mixed components of the second-order tensor **T**.

The tensor product **T** of two vectors in an orthonormal basis (e.g., Euclidean coordinate system) is an invariant (scalar). The invariant is independent of the coordinate system and has an intrinsic value in any coordinate transformations. In Newtonian mechanics, the mechanical work **W** that is created by the force vector **F** and path vector **x** does not change in any chosen coordinate system. This mechanical work $\mathbf{W} = \mathbf{F}.\mathbf{x}$ is called an invariant and has an intrinsic value of energy.

Given three arbitrary vectors **u**, **v**, and **w** in $\mathbf{R}^N$ and a scalar α in **R**, the tensor product of two vectors has the following properties (Klingbeil 1966):

- Distributive property

$$\mathbf{u}(\mathbf{v}+\mathbf{w}) = \mathbf{uv}+\mathbf{uw}$$
$$(\mathbf{u}+\mathbf{v})\mathbf{w} = \mathbf{uw}+\mathbf{vw}$$

- Associative property

$$(\alpha\,\mathbf{u})\mathbf{v} = \mathbf{u}(\alpha\,\mathbf{v}) = \alpha\,\mathbf{uv}$$

(b) Cross product

The cross product of two vectors **u** and **v** can be defined by a linear functional **T**.

$$\mathbf{T}: \mathbf{u}, \mathbf{v} \in \mathbf{R}^N \rightarrow \mathbf{T} = \mathbf{u} \times \mathbf{v} = u^i v^j (\mathbf{g}_i \times \mathbf{g}_j) \in \mathbf{R}^N \tag{2.81}$$

Obviously, the cross product **T** of two vectors is a vector (first-order tensor) of which the direction is perpendicular to the bases of $\mathbf{g}_i$ and $\mathbf{g}_j$.

Using the scalar triple product in Eq. (1.10), the cross product of the bases can be written as

$$\left(\mathbf{g}_i \times \mathbf{g}_j\right) = \varepsilon_{ijk}\sqrt{g}\,\mathbf{g}^k = \varepsilon_{ijk} J\,\mathbf{g}^k \tag{2.82}$$

where ε_{ijk} is the Levi-Civita permutation symbol; J is the Jacobian.

Thus, the cross product in Eq. (2.81) can be expressed as

$$\begin{aligned}\mathbf{T} = \mathbf{u} \times \mathbf{v} &\equiv T_k \mathbf{g}^k \\ &= \left(\varepsilon_{ijk}\sqrt{g}\,u^i v^j\right)\mathbf{g}^k = \left(\varepsilon_{ijk} J\,u^i v^j\right)\mathbf{g}^k\end{aligned} \tag{2.83}$$

The covariant component T_k of the first-order tensor **T** results from Eqs. (2.81), (2.82), and (2.83).

$$\mathbf{T} = u^i v^j \left(\mathbf{g}_i \times \mathbf{g}_j\right) = u^i v^j \left(\mathbf{g}_i \times \mathbf{g}_j\right).\mathbf{g}_k\, \mathbf{g}^k \equiv T_k \mathbf{g}^k$$
$$\Rightarrow T_k = u^i v^j \left(\mathbf{g}_i \times \mathbf{g}_j\right).\mathbf{g}_k = \varepsilon_{ijk}\sqrt{g}\, u^i v^j = \varepsilon_{ijk} J\, u^i v^j \tag{2.84}$$

The Levi-Civita permutation symbol (pseudo-tensor) can be defined as

$$\varepsilon_{ijk} = \begin{cases} +1 \text{ if } (i, j, k) \text{ is an even permutation;} \\ -1 \text{ if } (i, j, k) \text{ is an odd permutation;} \\ 0 \text{ if } i = j, \text{or } i = k; \text{or } j = k \end{cases} \tag{2.85}$$

Therefore,

$$\varepsilon_{ijk} = \begin{cases} \varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \text{ (even permutation);} \\ -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik} \text{ (odd permutation)} \end{cases} \tag{2.86}$$

The permutation symbol ε_{ijk} contains totally 27 elements ($N^n = 3^3$) for i, j, k ($n = 3$) in a 27-dimensional tensor space $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3$.

Note that the permutation symbol is used in Eq. (2.83) because the direction of the cross product vector is opposite if the dummy indices are interchanged with each other in Einstein summation convention (cf. Appendix A).

$$\sqrt{g}\,\mathbf{g}^k = J\,\mathbf{g}^k = \left(\mathbf{g}_i \times \mathbf{g}_j\right) = -\left(\mathbf{g}_j \times \mathbf{g}_i\right)$$
$$\Rightarrow \mathbf{g}^k = \frac{\varepsilon_{ijk}\left(\mathbf{g}_i \times \mathbf{g}_j\right)}{\sqrt{g}} = \frac{\varepsilon_{ijk}\left(\mathbf{g}_i \times \mathbf{g}_j\right)}{J} \tag{2.87}$$
$$\Rightarrow \left(\mathbf{g}_i \times \mathbf{g}_j\right) = \varepsilon_{ijk}\sqrt{g}\mathbf{g}^k = \varepsilon_{ijk} J \mathbf{g}^k$$

where J denotes the Jacobian.

2.3.5 *Rules of Tensor Calculations*

In order to carry out the tensor calculations, some fundamental rules must be taken into account in tensor calculus.

1. **Calculation of tensor components**

Let $\mathbf{T}$ be a second-order tensor that can be written in different tensor forms:

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j = T^i_j\mathbf{g}_i\mathbf{g}^j = T_{ij}\mathbf{g}^i\mathbf{g}^j = T_i^j\mathbf{g}_j\mathbf{g}^i \tag{2.88}$$

Multiplying the first row of Eq. (2.88) by the covariant basis $\mathbf{g}_k$, one obtains

$$T^{ij}(\mathbf{g}_k.\mathbf{g}_j)\mathbf{g}_i = T^{ij}g_{kj}\mathbf{g}_i = T^i_k\mathbf{g}_i$$
$$\Rightarrow T^i_k = T^{ij}g_{kj} \quad \text{for } j = 1, 2, \ldots, N \tag{2.89a}$$

Analogously, multiplying the second row of Eq. (2.88) by the contravariant basis $\mathbf{g}^k$, one obtains

$$T_{ij}(\mathbf{g}^k.\mathbf{g}^j)\mathbf{g}^i = T_{ij}g^{kj}\mathbf{g}^i = T^k_i\mathbf{g}^i$$
$$\Rightarrow T^k_i = T_{ij}g^{kj} \quad \text{for } j = 1, 2, \ldots, N \tag{2.89b}$$

Multiplying Eq. (2.89a) by g^{kj}, the contravariant tensor components result in

$$T^{ij} = T^i_k g^{kj} \quad \text{for } k = 1, 2, \ldots, N \tag{2.90a}$$

Multiplying Eq. (2.89b) by g_{kj}, one obtains the covariant tensor components

$$T_{ij} = T^k_i g_{kj} \quad \text{for } k = 1, 2, \ldots, N \tag{2.90b}$$

Substituting Eqs. (2.89a) and (2.90a), one obtains the contraction rules between the contravariant tensor components.

$$T^{ij} = T^{ip}g_{pk}g^{kj} \quad \text{for } k, p = 1, 2, \ldots, N \tag{2.91a}$$

Similarly, the contraction rules between the covariant tensor components result from substituting Eqs. (2.89b) and (2.90b).

$$T_{ij} = T_{ip}g^{pk}g_{kj} \quad \text{for } k, p = 1, 2, \ldots, N \tag{2.91b}$$

Analogously, the contraction rules between the mixed tensor components can be derived as

$$T^i_j = T^i_p g^{pk}g_{kj} \quad \text{for } k, p = 1, 2, \ldots, N \tag{2.92a}$$

Similarly, one obtains

$$T^j_i = T^p_i g_{pk}g^{kj} \quad \text{for } k, p = 1, 2, \ldots, N \tag{2.92b}$$

2. **Addition law**

Tensors of the same orders and types can be added together. The resulting tensor has the same order and type of the initial tensors. The tensor resulted from the addition of two covariant or contravariant tensors **A** and **B** can be calculated as

$$\begin{aligned}
\mathbf{C} = \mathbf{A} + \mathbf{B} &= \left(A_{ijk} + B_{ijk}\right)\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k = C_{ijk}\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k = \mathbf{B} + \mathbf{A} \\
&\Rightarrow C_{ijk} = A_{ijk} + B_{ijk} = B_{ijk} + A_{ijk}; \\
\mathbf{C} = \mathbf{A} + \mathbf{B} &= \left(A^{ijk} + B^{ijk}\right)\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k = C^{ijk}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k = \mathbf{B} + \mathbf{A} \\
&\Rightarrow C^{ijk} = A^{ijk} + B^{ijk} = B^{ijk} + A^{ijk}
\end{aligned} \tag{2.93}$$

Similarly, the tensor resulted from the addition of two mixed tensors **A** and **B** can be written as

$$\begin{aligned}
\mathbf{C} = \mathbf{A} + \mathbf{B} &= \left(A_{ijk}^{pq} + B_{ijk}^{pq}\right)\mathbf{g}_p\mathbf{g}_q\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k \\
&= C_{ijk}^{pq}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k = \mathbf{B} + \mathbf{A} \\
&\Rightarrow C_{ijk}^{pq} = A_{ijk}^{pq} + B_{ijk}^{pq} = B_{ijk}^{pq} + A_{ijk}^{pq}
\end{aligned} \tag{2.94}$$

Straightforwardly, the addition of tensors is commutative, as proved in Eqs. (2.93) and (2.94).

3. **Outer product**

On the contrary, the outer product can be carried out at tensors of different orders and types. The tensor components resulted from the outer product of two mixed tensors **A** and **B** can be calculated as

$$\begin{aligned}
\mathbf{AB} &= \left(A_{ij}^{pq}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^i\mathbf{g}^j\right)\left(B_{kl}^{rts}\mathbf{g}^k\mathbf{g}^l\mathbf{g}_r\mathbf{g}_t\mathbf{g}_s\right) \\
&= C_{ijkl}^{pqrst}\mathbf{g}_p\mathbf{g}_q\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k\mathbf{g}^l\mathbf{g}_r\mathbf{g}_t\mathbf{g}_s \neq \mathbf{BA} \\
&\Rightarrow C_{ijkl}^{pqrst} = A_{ij}^{pq}B_{kl}^{rts} = B_{kl}^{rts}A_{ij}^{pq}
\end{aligned} \tag{2.95}$$

The outer product of two tensors results a tensor with the order that equals the sum of the covariant and contravariant indices. The outer product is not commutative, but their tensor components are commutative, as shown in Eq. (2.95). In this example, the resulting ninth-order tensor is generated from the outer product of the mixed fourth-order tensor **A** and mixed fifth-order tensor **B**. Obviously, the outer product of tensors **A**, **B**, and **C** is associative, i.e., **A**(**BC**) = (**AB**)**C**.

4. **Contraction law**

The contraction operation can be only carried out at the mixed tensor types of different orders. The tensor contraction is operated in many contracting steps where the tensor order is shortened by eliminating the same covariant and contravariant indices of the tensor components.

We consider a mixed tensor of high orders. In this example, the mixed fifth-order tensor **A** of type (2, 3) can be transformed from the coordinates $\{u^i\}$ into the barred coordinates $\left\{\bar{u}^{\mathrm{i}}\right\}$. The transformed tensor components can be calculated according to the transformation law in Eq. (2.144).

$$\bar{A}^{ij}_{klm} = \frac{\partial \bar{u}^i}{\partial u^p}\frac{\partial \bar{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \bar{u}^k}\frac{\partial u^s}{\partial \bar{u}^l}\frac{\partial u^t}{\partial \bar{u}^m} A^{pq}_{rst} \tag{2.96}$$

Carrying out the first contraction of $\bar{A}$ in Eq. (2.96) at $l = i$, one obtains the tensor components

$$\begin{aligned}
\bar{A}^{ij}_{kim} &= \frac{\partial \bar{u}^i}{\partial u^p}\frac{\partial \bar{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \bar{u}^k}\frac{\partial u^s}{\partial \bar{u}^l}\frac{\partial u^t}{\partial \bar{u}^m} A^{pq}_{rst} \\
&= \frac{\partial \bar{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \bar{u}^k}\frac{\partial u^t}{\partial \bar{u}^m}\left(\frac{\partial \bar{u}^i}{\partial u^p}\frac{\partial u^s}{\partial \bar{u}^i}\right) A^{pq}_{rst} \\
&= \frac{\partial \bar{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \bar{u}^k}\frac{\partial u^t}{\partial \bar{u}^m}\delta^s_p A^{pq}_{rst} \\
&= \frac{\partial \bar{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \bar{u}^k}\frac{\partial u^t}{\partial \bar{u}^m} A^{pq}_{rpt} \\
\Leftrightarrow \bar{B}^j_{km} &= \frac{\partial \bar{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \bar{u}^k}\frac{\partial u^t}{\partial \bar{u}^m} B^q_{rt}
\end{aligned} \tag{2.97a}$$

As a result, the resulting tensor components B are the third-order tensor type after the first contraction of A at $s = p$:

$$A^{pq}_{rst}\delta^p_s = A^{pq}_{rpt} \equiv B^q_{rt} \tag{2.97b}$$

Further contracting the tensor components $\bar{B}$ in Eq. (2.97a) at $k = j$, one obtains

$$\begin{aligned}
\bar{B}^j_{jm} &= \frac{\partial u^t}{\partial \bar{u}^m}\left(\frac{\partial \bar{u}^j}{\partial u^q}\frac{\partial u^r}{\partial \bar{u}^j}\right) B^q_{rt} \\
&= \frac{\partial u^t}{\partial \bar{u}^m}\delta^r_q B^q_{rt} \\
&= \frac{\partial u^t}{\partial \bar{u}^m} B^q_{qt} \\
\Leftrightarrow \bar{C}_m &= \frac{\partial u^t}{\partial \bar{u}^m} C_t
\end{aligned} \tag{2.98a}$$

As a result, the resulting tensor components C are the first-order tensor type after the second contraction of B in Eq. (2.98a) at $r = q$:

$$B^q_{rt}\delta^q_r = B^q_{qt} \equiv C_t \tag{2.98b}$$

5. **Inner product**

The inner product of tensors comprises two basic operations of the outer product and at least one contraction of tensors. As an example, the outer product of two third-order tensors **A** and **B** results in a sixth-order tensor.

$$\begin{aligned}\mathbf{AB} &= \left(A_q^{mp}\mathbf{g}_m\mathbf{g}_p\mathbf{g}^q\right)\left(B^r_{st}\mathbf{g}_r\mathbf{g}^s\mathbf{g}^t\right) \\ &= A_q^{mp}B^r_{st}\mathbf{g}^q\mathbf{g}_r\mathbf{g}_m\mathbf{g}_p\mathbf{g}^s\mathbf{g}^t\end{aligned} \tag{2.99}$$

Using the first tensor contraction in Eq. (2.99) at $r = q$, one obtains the resulting fourth-order tensor components of the inner product.

$$A_q^{mp}B^r_{st}\delta^r_q = A_q^{mp}B^q_{st} \equiv C^{mp}_{st} \tag{2.100}$$

Similarly, using the second tensor contraction law in Eq. (2.100) at $p = s$, the resulting second-order tensor components result in

$$C^{mp}_{st}\delta^p_s = C^{ms}_{st} \equiv D^m_t \tag{2.101}$$

Finally, applying the third tensor contraction law to Eq. (2.101) at $m = t$, the resulting tensor component is an invariant (zeroth-order tensor).

$$D^m_t\delta^m_t = D^t_t \equiv D \tag{2.102}$$

In another approach, one can calculate the tensor components of the inner product of two contravariant tensors **A** and **B** multiplying by the metric tensor.

$$A^{ijk}B^{lm} \rightarrow A^{ijk}B^{lm}g_{lk} \tag{2.103}$$

Using Eq. (2.89a), one obtains the resulting tensor components

$$B^{lm}g_{lk} = B^m_k \tag{2.104}$$

Substituting Eq. (2.104) into Eq. (2.103) and using the tensor contraction law, one obtains the resulting tensor components

$$C^{ijm} \equiv A^{ijk}B^m_k = B^m_k A^{ijk} \tag{2.105}$$

Equation (2.105) denotes that the inner product of the tensor components is commutative.

6. **Indices law**

Using the metric tensors, the operation of moving indices enables changing indices of the tensor components from the upper into lower positions and vice versa. Multiplying a tensor component by the metric tensor components, the lower index (covariant index) is moved into the upper index (contravariant index) and vice versa.

– Moving covariant indices i, j to the upper position:

$$A^k_{ij} \rightarrow A^k_{ij}g^{il} = A^{kl}_j \rightarrow A^{kl}_j g^{jm} = A^{klm} \tag{2.106a}$$

– Moving contravariant indices i, j to the lower position:

$$A_k^{ij} \to A_k^{ij} g_{jl} = A_{kl}^{i} \to A_{kl}^{i} g_{im} = A_{klm} \tag{2.106b}$$

7. **Quotient law**

The quotient law of tensors postulates that if the tensor product of **AB** and **B** are tensors, **A** must be a tensor.

$$(\mathbf{AB} = \mathbf{C} \therefore \text{tensor}) \cap (\mathbf{B} \therefore \text{tensor}) \Rightarrow (\mathbf{A} \therefore \text{tensor}) \tag{2.107a}$$

Proof Using the contraction law, the barred components in the transformation coordinates $\{\bar{u}^{\mathrm{i}}\}$ of the tensor product **AB** result in

$$A_{ij}^{k} B_{k}^{il} = C_{j}^{l} \Rightarrow \bar{A}_{ij}^{k} \bar{B}_{k}^{il} = \bar{C}_{j}^{l} \tag{2.107b}$$

According to the transformation law (2.144), the transformed components in the coordinates $\{\bar{u}^{\mathrm{i}}\}$ of the tensors **B** and **C** can be calculated as

$$\begin{aligned} \bar{B}_{k}^{il} &= B_{p}^{mn} \frac{\partial \bar{u}^{i}}{\partial u^{m}} \frac{\partial \bar{u}^{l}}{\partial u^{n}} \frac{\partial u^{p}}{\partial \bar{u}^{k}}; \\ \bar{C}_{j}^{l} &= C_{q}^{n} \frac{\partial \bar{u}^{l}}{\partial u^{n}} \frac{\partial u^{q}}{\partial \bar{u}^{j}} \end{aligned} \tag{2.107c}$$

Substituting Eq. (2.107c) into Eq. (2.107b) and using the contraction law, one obtains

$$\begin{aligned} \bar{A}_{ij}^{k} \left(B_{p}^{mn} \frac{\partial \bar{u}^{i}}{\partial u^{m}} \frac{\partial \bar{u}^{l}}{\partial u^{n}} \frac{\partial u^{p}}{\partial \bar{u}^{k}} \right) &= C_{q}^{n} \frac{\partial \bar{u}^{l}}{\partial u^{n}} \frac{\partial u^{q}}{\partial \bar{u}^{j}} \\ &= \left(A_{mq}^{p} B_{p}^{mn} \right) \frac{\partial \bar{u}^{l}}{\partial u^{n}} \frac{\partial u^{q}}{\partial \bar{u}^{j}} \end{aligned} \tag{2.107d}$$

Rearranging the terms of Eq. (2.107d), one obtains

$$\begin{aligned} &\left(\bar{A}_{ij}^{k} \frac{\partial \bar{u}^{i}}{\partial u^{m}} \frac{\partial u^{p}}{\partial \bar{u}^{k}} - A_{mq}^{p} \frac{\partial u^{q}}{\partial \bar{u}^{j}} \right) \frac{\partial \bar{u}^{l}}{\partial u^{n}} B_{p}^{mn} = 0 \\ &\Rightarrow \forall B_{p}^{mn}; \frac{\partial \bar{u}^{l}}{\partial u^{n}} \neq 0 : \bar{A}_{ij}^{k} \frac{\partial \bar{u}^{i}}{\partial u^{m}} \frac{\partial u^{p}}{\partial \bar{u}^{k}} = A_{mq}^{p} \frac{\partial u^{q}}{\partial \bar{u}^{j}} \end{aligned} \tag{2.107e}$$

Applying the inner product by $\left(\frac{\partial u^{r}}{\partial \bar{u}^{i}} \frac{\partial \bar{u}^{k}}{\partial u^{s}} \right)$ to Eq. (2.107e), one obtains the barred components of **A** at $r = m$ and $s = p$.

$$\begin{aligned}\bar{A}^k_{ij}\delta^r_m\delta^p_s &= A^p_{mq}\frac{\partial u^q}{\partial \bar{u}^j}\left(\frac{\partial u^r}{\partial \bar{u}^i}\frac{\partial \bar{u}^k}{\partial u^s}\right)\\ \Rightarrow \bar{A}^k_{ij} &= A^p_{mq}\frac{\partial \bar{u}^k}{\partial u^p}\frac{\partial u^m}{\partial \bar{u}^i}\frac{\partial u^q}{\partial \bar{u}^j}\end{aligned} \tag{2.107f}$$

Equation (2.107f) proves that **A** is a mixed third-order tensor of type (1, 2), cf. Eq. (2.107c).

8. **Symmetric tensors**

Tensor **T** is called symmetric in the given basis if two covariant or contravariant indices of the tensor component can be interchanged without changing the tensor component value.

$$\begin{aligned}T_{ij} &= T_{ji}\text{: symmetric in } i \text{ and } j\\ T^{ij} &= T^{ji}\text{: symmetric in } i \text{ and } j\\ T^{ijk}_{pq} &= T^{ikj}_{pq}\text{: symmetric in } j \text{ and } k\\ T^{ijk}_{pq} &= T^{ijk}_{qp}\text{: symmetric in } p \text{ and } q\end{aligned} \tag{2.108}$$

In the case of a second-order tensor, the tensor **T** is symmetric if **T** equals its transpose.

$$\mathbf{T} = \mathbf{T}^T \tag{2.109}$$

9. **Skew-symmetric tensors**

The sign of the tensor component is opposite if a pair of the covariant or contravariant indices are interchanged with each other. In this case, the tensor is skew-symmetric (antisymmetric).

Tensor **T** is defined as a skew-symmetric tensor (antisymmetric) if

$$\begin{aligned}T_{ij} &= -T_{ji}\text{: skew-symmetric in } i \text{ and } j\\ T^{ij} &= -T^{ji}\text{: skew-symmetric in } i \text{ and } j\\ T^{ijk}_{pq} &= -T^{ikj}_{pq}\text{: skew-symmetric in } j \text{ and } k\\ T^{ijk}_{pq} &= -T^{ijk}_{qp}\text{: skew-symmetric in } p \text{ and } q\end{aligned} \tag{2.110}$$

In the case of a second-order tensor, the tensor **T** is skew-symmetric if **T** is opposite to its transpose.

$$\mathbf{T} = -\mathbf{T}^T \tag{2.111}$$

An arbitrary tensor $\mathbf{T}$ can be generally decomposed into the symmetric and skew-symmetric tensors:

$$\mathbf{T} = \tfrac{1}{2}\left(\mathbf{T} + \mathbf{T}^T\right) + \tfrac{1}{2}\left(\mathbf{T} - \mathbf{T}^T\right) \equiv \mathbf{T}_{\mathrm{sym}} + \mathbf{T}_{\mathrm{skew}} \tag{2.112}$$

Proof

- The first tensor $\mathbf{T}_{\mathrm{sym}}$ is symmetric:

$$\mathbf{T}_{\mathrm{sym}} \equiv \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^T\right) = \mathbf{T}_{\mathrm{sym}}^T \qquad \text{(qed)}$$

- The second tensor $\mathbf{T}_{\mathrm{skew}}$ is skew-symmetric:

$$\mathbf{T}_{\mathrm{skew}} \equiv \frac{1}{2}\left(\mathbf{T} - \mathbf{T}^T\right) = -\mathbf{T}_{\mathrm{skew}}^T \qquad \text{(qed)}$$

2.4 Coordinate Transformations

Tensors are tuples of independent coordinates in a finite multifold N-dimensional tensor space ($\mathbf{R}^N \times \cdots \times \mathbf{R}^N$). The tensor describes physical states generally depending on different variables (dimensions). Each physical state can be defined as the point P($u^1,\ldots, u^N$) with N coordinates of u^i. By changing the variables, such as time, locations, and physical characteristics (e.g., pressure, temperature, density, velocity), the physical state point varies in the multifold N-dimensional space.

The tensor does not change itself and is independent in any coordinate system. However, its components change in the new basis by the coordinate transformation since the basis changes as the coordinate system varies. In this case, applications of tensor analysis have been used to describe the transformation between two general curvilinear coordinate systems in the multifold N-dimensional spaces. Hence, tensors are a very useful tool applied to the coordinate transformations in the multifold N-dimensional tensor spaces. High-order tensors can be generated by a multilinear map between two multifold N-dimensional spaces (cf. Sect. 2.2). Their components change in the relating bases by the coordinate transformations, as displayed in Fig. 2.5.

In the following section, the relations between the tensor components in different curvilinear coordinates of the finite N-dimensional spaces will be discussed.

2.4.1 Transformation in the Orthonormal Coordinates

The simple coordinate transformation of rotation between the orthonormal coordinates x_i and u_j in Euclidean coordinate system is carried out.

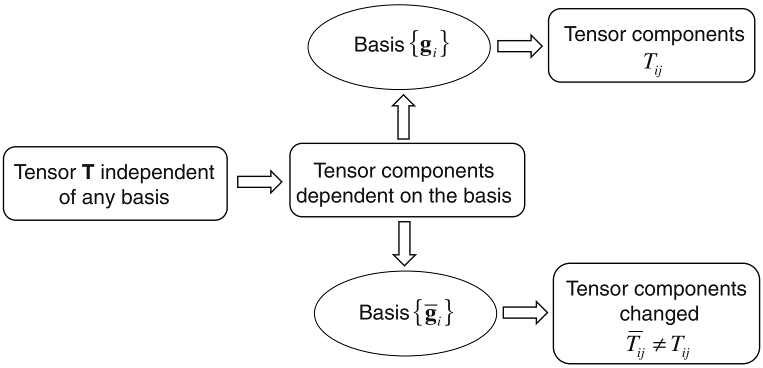

Fig. 2.5 Tensor and tensor components in different bases

An arbitrary vector **r** (first-order tensor) can be written in both coordinate systems:

$$\begin{aligned} \mathbf{r} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \equiv x_i\mathbf{e}_i \\ &= u_1\mathbf{g}_1 + u_2\mathbf{g}_2 \equiv u_j\mathbf{g}_j \end{aligned} \tag{2.113}$$

The vector components in the coordinate u_j can be calculated in

$$\begin{cases} u_1 = \cos\theta_{11}x_1 + \cos\theta_{12}x_2 \\ u_2 = \cos\theta_{21}x_1 + \cos\theta_{22}x_2 \end{cases} \tag{2.114}$$

Thus,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} \\ \cos\theta_{21} & \cos\theta_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Leftrightarrow \mathbf{u} = \mathbf{Tx} \tag{2.115}$$

where **T** is the transformation matrix.

Setting $\theta_{11} = \theta$, one obtains

$$\begin{aligned} &\cos\theta_{11} = \cos\theta;\ \cos\theta_{12} = \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta \\ &\cos\theta_{21} = \cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta;\ \cos\theta_{22} = \cos\theta \end{aligned}$$

Therefore, the transformation matrix **T** becomes

$$\mathbf{T} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{12} \\ \cos\theta_{21} & \cos\theta_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.116}$$

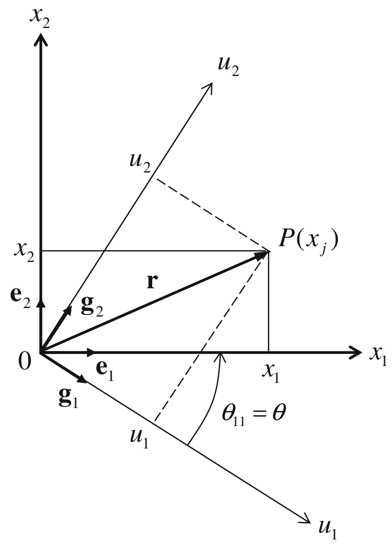

Fig. 2.6 Two-dimensional coordinate transformation of rotation

The transformed coordinates can be computed by the transformation T:

$$T : \mathbf{x} \rightarrow \mathbf{u} = \mathbf{Tx} : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.117}$$

where θ is the rotation angle of the rotating coordinates u_j.

Transforming back Eq. (2.117), one obtains the coordinates x_i (Fig. 2.6)

$$T^{-1} : \mathbf{u} \rightarrow \mathbf{x} = \mathbf{T}^{-1}\mathbf{u} : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.118}$$

The vector component on the basis is obtained multiplying Eq. (2.113) by the relating basis $\mathbf{e}_i$ or $\mathbf{g}_j$.

$$\begin{aligned} x_\mathrm{i} &= \mathbf{r}.\mathbf{e}_\mathrm{i}; \quad \mathrm{i} = 1, 2 \\ u_\mathrm{j} &= \mathbf{r}.\mathbf{g}_\mathrm{j}; \quad \mathrm{j} = 1, 2 \end{aligned} \tag{2.119}$$

Substituting Eq. (2.119) into Eq. (2.117), one obtains the transformation matrix between two coordinate systems.

$$\begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} \Leftrightarrow \mathbf{g} = \mathbf{T}.\mathbf{e} \tag{2.120}$$

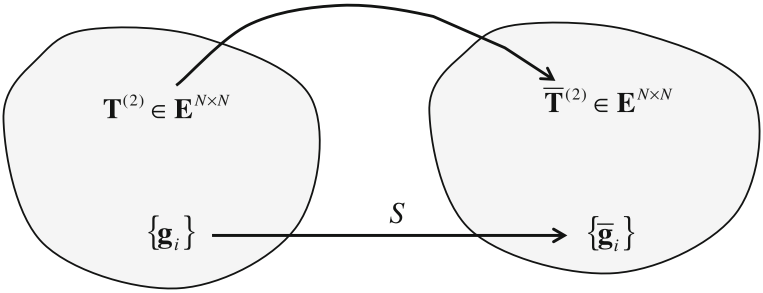

Fig. 2.7 Basis transformation of general curvilinear coordinates in $\mathbf{E}^N$

Similarly,

$$\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} . \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} \Leftrightarrow \mathbf{e} = \mathbf{T}^{-1}.\mathbf{g} \tag{2.121}$$

2.4.2 Transformation of Curvilinear Coordinates in E^N

In the following section, second-order tensors **T** are used in the transformation of general curvilinear coordinates in Euclidean space $\mathbf{E}^{\mathrm{N}}$, as shown in Fig. 2.7.

The basis $\mathbf{g}_i$ of the curvilinear coordinate $\{u^i\}$ can be transformed into the new basis $\bar{\mathbf{g}}_i$ of the curvilinear coordinate $\{\bar{u}^i\}$ using the linear transformation **S**. The new covariant basis can be rewritten as a linear combination of the old basis.

$$S : \mathbf{g}_i \to \bar{\mathbf{g}}_i = S_i^j \mathbf{g}_j \Leftrightarrow \bar{\mathbf{G}} = \mathbf{G}\mathbf{S} \tag{2.122}$$

where S_i^j are the mixed transformation components of the second-order tensor **S**.

The old covariant basis results can be calculated as

$$\mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^j} = \frac{\partial \mathbf{r}}{\partial \bar{u}^i} \frac{\partial \bar{u}^i}{\partial u^j} \equiv \bar{\mathbf{g}}_i \left(S^{-1}\right)^i_j \Rightarrow \left(S^{-1}\right)^i_j \equiv \frac{\partial \bar{u}^i}{\partial u^j} \tag{2.123}$$

Inversing the basis matrix in Eq. (2.122), the new contravariant basis can be calculated as

$$\bar{\mathbf{G}}^{-1} = (\mathbf{G}\mathbf{S})^{-1} = \mathbf{S}^{-1}\mathbf{G}^{-1} \Rightarrow \bar{\mathbf{g}}^i = \left(S^{-1}\right)^i_j \mathbf{g}^j \tag{2.124}$$

Multiplying Eq. (2.124) by the linear transformation **S**, the old contravariant basis results in

$$\mathbf{G}^{-1} = \mathbf{S}\bar{\mathbf{G}}^{-1} \Rightarrow \mathbf{g}^i = S^i_j \bar{\mathbf{g}}^j \tag{2.125}$$

According to Eqs. (2.11) and (2.122), the new covariant basis can be calculated as

$$\bar{\mathbf{g}}_j = \frac{\partial \mathbf{r}}{\partial \bar{u}^j} = \frac{\partial \mathbf{r}}{\partial u^k}\frac{\partial u^k}{\partial \bar{u}^j} \equiv \mathbf{g}_k S^k_j \Rightarrow S^k_j \equiv \frac{\partial u^k}{\partial \bar{u}^j} \tag{2.126a}$$

Combining Eqs. (2.123), (2.124), and (2.126a) and using the differentiation chain rule, one obtains the relation of the mixed transformation components between two general curvilinear coordinates in Euclidean space $\mathbf{E}^{\mathrm{N}}$.

$$\begin{aligned} \bar{\mathbf{g}}_j.\bar{\mathbf{g}}^i &= \bar{\mathbf{g}}^i.\bar{\mathbf{g}}_j = \left(S^{-1}\right)^i_l S^k_j \mathbf{g}^l.\mathbf{g}_k = \left(S^{-1}\right)^i_l S^k_j \delta^l_k \\ &= \left(S^{-1}\right)^i_k S^k_j = \frac{\partial \bar{u}^i}{\partial u^k}\frac{\partial u^k}{\partial \bar{u}^j} = \frac{\partial \bar{u}^i}{\partial \bar{u}^j} = \delta^i_j \\ &\Leftrightarrow \mathbf{S}^{-1}\mathbf{S} = \mathbf{I} \end{aligned} \tag{2.126b}$$

Therefore, the transformation tensor **S** can be written as

$$\mathbf{S} = \begin{bmatrix} \frac{\partial u^1}{\partial \bar{u}^1} & \frac{\partial u^1}{\partial \bar{u}^2} & \cdot & \frac{\partial u^1}{\partial \bar{u}^N} \\ \frac{\partial u^2}{\partial \bar{u}^1} & \frac{\partial u^2}{\partial \bar{u}^2} & \cdot & \frac{\partial u^2}{\partial \bar{u}^N} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial u^N}{\partial \bar{u}^1} & \frac{\partial u^N}{\partial \bar{u}^2} & \cdot & \frac{\partial u^N}{\partial \bar{u}^N} \end{bmatrix} \in \mathbf{R}^N \times \mathbf{R}^N \tag{2.127a}$$

The transformation tensor **S** in Eq. (2.127a) is identical to the Jacobian matrix between two coordinate systems $\{u^i\}$ and $\{\bar{u}^i\}$.

Inverting the transformation tensor **S**, the back transformation results in

$$\mathbf{S}^{-1} = \begin{bmatrix} \frac{\partial \bar{u}^1}{\partial u^1} & \frac{\partial \bar{u}^1}{\partial u^2} & \cdot & \frac{\partial \bar{u}^1}{\partial u^N} \\ \frac{\partial \bar{u}^2}{\partial u^1} & \frac{\partial \bar{u}^2}{\partial u^2} & \cdot & \frac{\partial \bar{u}^2}{\partial u^N} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial \bar{u}^N}{\partial u^1} & \frac{\partial \bar{u}^N}{\partial u^2} & \cdot & \frac{\partial \bar{u}^N}{\partial u^N} \end{bmatrix} \in \mathbf{R}^N \times \mathbf{R}^N \tag{2.127b}$$

The relation between the new and old components of an arbitrary vector **v** (first-order tensor) is similarly given in the coordinate transformation S according to Eqs. (2.122) and (2.124).

$$\begin{aligned} \bar{v}_i &= \bar{\mathbf{g}}_{\mathrm{i}}.\mathbf{v} = S^j_i \mathbf{g}_j.\left(v_j \mathbf{g}^j\right) = S^j_i v_j \\ \bar{v}^i &= \bar{\mathbf{g}}^{\mathrm{i}}.\mathbf{v} = \left(S^{-1}\right)^i_j \mathbf{g}^j.\left(v^j \mathbf{g}_j\right) = \left(S^{-1}\right)^i_j v^j \end{aligned} \tag{2.128}$$

Using Eq. (2.74), the relation of the components of the second-order tensor T can be derived.

- Covariant metric tensor components:

$$\begin{aligned}\bar{T}_{ij} &= \bar{\mathbf{g}}_i.\mathbf{T}.\bar{\mathbf{g}}_j = \left(S_i^k \mathbf{g}_k\right).\mathbf{T}.\left(S_j^l \mathbf{g}_l\right) \\ &= S_i^k S_j^l \left(\mathbf{g}_k.\mathbf{T}.\mathbf{g}_l\right) = S_i^k S_j^l T_{kl}\end{aligned} \tag{2.129}$$

- Contravariant metric tensor components:

$$\begin{aligned}\bar{T}^{ij} &= \bar{\mathbf{g}}^i.\mathbf{T}.\bar{\mathbf{g}}^j = \left(S^{-1}\right)_k^i \mathbf{g}^k.\mathbf{T}.\left(S^{-1}\right)_l^j \mathbf{g}^l \\ &= \left(S^{-1}\right)_k^i \left(S^{-1}\right)_l^j \left(\mathbf{g}^k.\mathbf{T}.\mathbf{g}^l\right) = \left(S^{-1}\right)_k^i \left(S^{-1}\right)_l^j T^{kl}\end{aligned} \tag{2.130}$$

- Mixed metric tensor components:

$$\begin{aligned}\bar{T}_{j.}^{i} &= \bar{\mathbf{g}}_j.\mathbf{T}.\bar{\mathbf{g}}^i = \left(S_j^k \mathbf{g}_k\right).\mathbf{T}.\left(S^{-1}\right)_l^i \mathbf{g}^1 \\ &= S_j^k \left(S^{-1}\right)_l^i \left(\mathbf{g}_k.\mathbf{T}.\mathbf{g}^l\right) = S_j^k \left(S^{-1}\right)_l^i T_{k.}^{l}\end{aligned} \tag{2.131}$$

Note that in Eq. (2.131), the dot *after* the lower index indicates the position of the basis of the upper index locating *after* the tensor **T**. In this case, the tensor **T** is located between the upper basis $\mathbf{g}^i$ and lower basis $\mathbf{g}_j$.

$$\begin{aligned}\bar{T}_{.j}^{i} &= \bar{\mathbf{g}}^i.\mathbf{T}.\bar{\mathbf{g}}_j = \left(S^{-1}\right)_k^i \mathbf{g}^k.\mathbf{T}.\left(S_j^l \mathbf{g}_l\right) \\ &= \left(S^{-1}\right)_k^i S_j^l \left(\mathbf{g}^k.\mathbf{T}.\mathbf{g}_l\right) = \left(S^{-1}\right)_k^i S_j^l T_{.l}^{k}\end{aligned} \tag{2.132}$$

Note that in Eq. (2.132), the dot *before* the lower index indicates the position of the basis of the upper index locating *in front of* the tensor **T**. In this case, the tensor **T** is located between the upper basis $\mathbf{g}^i$ and lower basis $\mathbf{g}_j$.

2.4.3 Examples of Coordinate Transformations

(a) **Cylindrical Coordinates**

The transformation S from Cartesian $\{u^i\}$ to cylindrical coordinates $\{\bar{u}^i\}$:

$$S:\begin{cases} u^1 = r\cos\theta \equiv \bar{u}^1 \cos\bar{u}^2 \\ u^2 = r\sin\theta \equiv \bar{u}^1 \sin\bar{u}^2 \\ u^3 = z = \bar{u}^3 \end{cases}$$

The covariant transformation matrix **S** can be calculated as

$$\mathbf{S}=\begin{bmatrix}\dfrac{\partial u^1}{\partial \bar{u}^1} & \dfrac{\partial u^1}{\partial \bar{u}^2} & \dfrac{\partial u^1}{\partial \bar{u}^3}\\ \dfrac{\partial u^2}{\partial \bar{u}^1} & \dfrac{\partial u^2}{\partial \bar{u}^2} & \dfrac{\partial u^2}{\partial \bar{u}^3}\\ \dfrac{\partial u^3}{\partial \bar{u}^1} & \dfrac{\partial u^3}{\partial \bar{u}^2} & \dfrac{\partial u^3}{\partial \bar{u}^3}\end{bmatrix}=\begin{pmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

The determinant of **S** is called the Jacobian J.

$$|\mathbf{S}|=\begin{vmatrix}\dfrac{\partial u^1}{\partial \bar{u}^1} & \dfrac{\partial u^1}{\partial \bar{u}^2} & \dfrac{\partial u^1}{\partial \bar{u}^3}\\ \dfrac{\partial u^2}{\partial \bar{u}^1} & \dfrac{\partial u^2}{\partial \bar{u}^2} & \dfrac{\partial u^2}{\partial \bar{u}^3}\\ \dfrac{\partial u^3}{\partial \bar{u}^1} & \dfrac{\partial u^3}{\partial \bar{u}^2} & \dfrac{\partial u^3}{\partial \bar{u}^3}\end{vmatrix}\equiv J=\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=r$$

The contravariant transformation matrix $\mathbf{S}^{-1}$ results from the inversion of the covariant matrix **S**.

$$\mathbf{S}^{-1}=\begin{bmatrix}\dfrac{\partial \bar{u}^1}{\partial u^1} & \dfrac{\partial \bar{u}^1}{\partial u^2} & \dfrac{\partial \bar{u}^1}{\partial u^3}\\ \dfrac{\partial \bar{u}^2}{\partial u^1} & \dfrac{\partial \bar{u}^2}{\partial u^2} & \dfrac{\partial \bar{u}^2}{\partial u^3}\\ \dfrac{\partial \bar{u}^3}{\partial u^1} & \dfrac{\partial \bar{u}^3}{\partial u^2} & \dfrac{\partial \bar{u}^3}{\partial u^3}\end{bmatrix}=\frac{1}{r}\begin{pmatrix}r\cos\theta & r\sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & r\end{pmatrix}$$

(b) Spherical Coordinates

The transformation S from Cartesian $\{u^i\}$ to spherical coordinates $\{\bar{u}^i\}$:

$$S:\begin{cases}u^1=\rho\sin\phi\cos\theta\equiv\bar{u}^1\sin\bar{u}^2\cos\bar{u}^3\\ u^2=\rho\sin\phi\sin\theta\equiv\bar{u}^1\sin\bar{u}^2\cos\bar{u}^3\\ u^3=\rho\cos\phi=\bar{u}^1\cos\bar{u}^2\end{cases}$$

The covariant transformation matrix **S** can be calculated as

$$\mathbf{S}=\begin{bmatrix}\dfrac{\partial u^1}{\partial \bar{u}^1} & \dfrac{\partial u^1}{\partial \bar{u}^2} & \dfrac{\partial u^1}{\partial \bar{u}^3}\\ \dfrac{\partial u^2}{\partial \bar{u}^1} & \dfrac{\partial u^2}{\partial \bar{u}^2} & \dfrac{\partial u^2}{\partial \bar{u}^3}\\ \dfrac{\partial u^3}{\partial \bar{u}^1} & \dfrac{\partial u^3}{\partial \bar{u}^2} & \dfrac{\partial u^3}{\partial \bar{u}^3}\end{bmatrix}=\begin{pmatrix}\sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta\\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta\\ \cos\phi & -\rho\sin\phi & 0\end{pmatrix}$$

The determinant of **S** is called the Jacobian J.

$$|\mathbf{S}| = \begin{vmatrix} \dfrac{\partial u^1}{\partial \bar{u}^1} & \dfrac{\partial u^1}{\partial \bar{u}^2} & \dfrac{\partial u^1}{\partial \bar{u}^3} \\ \dfrac{\partial u^2}{\partial \bar{u}^1} & \dfrac{\partial u^2}{\partial \bar{u}^2} & \dfrac{\partial u^2}{\partial \bar{u}^3} \\ \dfrac{\partial u^3}{\partial \bar{u}^1} & \dfrac{\partial u^3}{\partial \bar{u}^2} & \dfrac{\partial u^3}{\partial \bar{u}^3} \end{vmatrix} \equiv J$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi$$

The contravariant transformation matrix results from the inversion of the matrix **S**.

$$\mathbf{S}^{-1} = \begin{bmatrix} \dfrac{\partial \bar{u}^1}{\partial u^1} & \dfrac{\partial \bar{u}^1}{\partial u^2} & \dfrac{\partial \bar{u}^1}{\partial u^3} \\ \dfrac{\partial \bar{u}^2}{\partial u^1} & \dfrac{\partial \bar{u}^2}{\partial u^2} & \dfrac{\partial \bar{u}^2}{\partial u^3} \\ \dfrac{\partial \bar{u}^3}{\partial u^1} & \dfrac{\partial \bar{u}^3}{\partial u^2} & \dfrac{\partial \bar{u}^3}{\partial u^3} \end{bmatrix} = \frac{1}{\rho}\begin{pmatrix} \rho\sin\phi\cos\theta & \rho\sin\phi\sin\theta & \rho\cos\phi \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\left(\dfrac{\sin\theta}{\sin\phi}\right) & \left(\dfrac{\cos\theta}{\sin\phi}\right) & 0 \end{pmatrix}$$

2.4.4 Transformation of Curvilinear Coordinates in R^N

In the following section, second-order tensors **T** will be used in the transformation of general curvilinear coordinates in Riemannian manifold $\mathbf{R}^{\mathrm{N}}$, as shown in Fig. 2.8. In Riemannian manifold, the bases $\mathbf{g}_i$ and $\bar{\mathbf{g}}_i$ of the curvilinear coordinates u^i and $\bar{u}^i$ do not exist any longer. Instead of the metric coefficients, the transformation coefficients that depend on the relating coordinates have been used in Riemannian manifold (Klingbeil 1966).

The new barred curvilinear coordinate $\bar{u}^i$ is a function of the old curvilinear coordinates $u^j, j = 1, 2,\ldots, N$. Therefore, it can be written in a linear function of u^j.

$$\begin{aligned} \bar{u}^i &= \bar{u}^i(u^1,\ldots,u^N) \\ &\Rightarrow \bar{u}^i = \bar{a}^i_j u^j \;\text{ for } j = 1,2,\ldots,N \\ &\Rightarrow d\bar{u}^i = \bar{a}^i_j du^j \;\text{ for } j = 1,2,\ldots,N \end{aligned} \tag{2.133}$$

where $\bar{a}^{\mathrm{i}}_{\mathrm{j}}$ is the transformation coefficient in the coordinate transformation S.

Using the chain rule of differentiation, one obtains

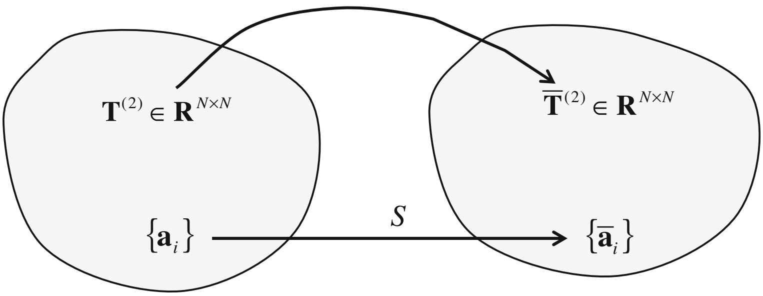

Fig. 2.8 Basis transformation of general curvilinear coordinates in R^N

$$\begin{aligned} \mathrm{d}\bar{u}^i &= \frac{\partial \bar{u}^i}{\partial u^j}\mathrm{d}u^j = \bar{a}^i_j \mathrm{d}u^j \\ &\Rightarrow \bar{a}^i_j = \frac{\partial \bar{u}^i}{\partial u^j} \end{aligned} \tag{2.134}$$

Analogously, the transformation coefficients of the back transformation result in

$$\begin{aligned} du^i &= \frac{\partial u^i}{\partial \bar{u}^j} d\bar{u}^j = \underline{a}^i_j d\bar{u}^j \\ &\Rightarrow \underline{a}^i_j = \frac{\partial u^i}{\partial \bar{u}^j} \end{aligned} \tag{2.135}$$

Combining Eqs. (2.134) and (2.135) and using the Kronecker delta, one obtains the relation between the transformation coefficients

$$\begin{aligned} \underline{a}^i_j \bar{a}^j_k &= \frac{\partial u^i}{\partial \bar{u}^j}\frac{\partial \bar{u}^j}{\partial u^k} = \frac{\partial u^i}{\partial u^k} = \delta^i_k \\ &\Leftrightarrow \left(\underline{a}^i_j\right)\left(\bar{a}^j_k\right) = \left(\bar{a}^j_k\right)\left(\underline{a}^i_j\right) = \mathbf{I} \end{aligned} \tag{2.136}$$

The relation of the second-order tensor components between the new and old curvilinear coordinates can be calculated using Eq. (2.133).

$$\begin{aligned} T^{(2)} &= \bar{T}_{kl}\bar{u}^k\bar{u}^l = \left(\bar{T}_{kl}\bar{a}^k_i\bar{a}^l_j\right)u^i u^j \\ &\equiv T_{ij}u^i u^j \end{aligned} \tag{2.137}$$

Thus,

$$\begin{aligned} T_{ij} &= \bar{a}^k_i \bar{a}^l_j \bar{T}_{kl} = \frac{\partial \bar{u}^k}{\partial u^i}\frac{\partial \bar{u}^l}{\partial u^j}\bar{T}_{kl}; \\ \bar{T}_{ij} &= \underline{a}^k_i \underline{a}^l_j T_{kl} = \frac{\partial u^k}{\partial \bar{u}^i}\frac{\partial u^l}{\partial \bar{u}^j}T_{kl} \end{aligned} \tag{2.138}$$

In the same way, the covariant, contravariant, and mixed components of the second-order tensor **T** between both coordinates in the transformation can be derived as:

- Covariant tensor components:

$$T_{ij} = \bar{a}_i^k \bar{a}_j^l \bar{T}_{kl} \Leftrightarrow \bar{T}_{ij} = \underline{a}_i^k \underline{a}_j^l T_{kl} \tag{2.139}$$

- Contravariant tensor components:

$$T^{ij} = \underline{a}_k^i \underline{a}_l^j \bar{T}^{kl} \Leftrightarrow \bar{T}^{ij} = \bar{a}_k^i \bar{a}_l^j T^{kl} \tag{2.140}$$

- Mixed tensor components:

$$T_{i.}^{j} = \bar{a}_i^k \underline{a}_l^j \bar{T}_{k.}^{l} \Leftrightarrow \bar{T}_{i.}^{j} = \underline{a}_i^k \bar{a}_l^j T_{k.}^{l} \tag{2.141}$$

$$T_{.j}^{i} = \underline{a}_k^i \bar{a}_j^l \bar{T}_{.l}^{k} \Leftrightarrow \bar{T}_{.j}^{i} = \bar{a}_k^i \underline{a}_j^l T_{.l}^{k} \tag{2.142}$$

Generally, the transformation coefficients of high-order tensors can be alternatively computed as

$$\begin{aligned} T_{ij}^{klm} &= \left(\frac{\partial u^k}{\partial \bar{u}^p} \frac{\partial u^l}{\partial \bar{u}^q} \frac{\partial u^m}{\partial \bar{u}^r} \right) \left(\frac{\partial \bar{u}^s}{\partial u^i} \frac{\partial \bar{u}^t}{\partial u^j} \right) \bar{T}_{st}^{pqr} \\ &= \left(\underline{a}_p^k \underline{a}_q^l \underline{a}_r^m \right) \left(\bar{a}_i^s \bar{a}_j^t \right) \bar{T}_{st}^{pqr} \end{aligned} \tag{2.143}$$

Therefore,

$$\begin{aligned} \bar{T}_{ij}^{klm} &= \left(\frac{\partial \bar{u}^k}{\partial u^p} \frac{\partial \bar{u}^l}{\partial u^q} \frac{\partial \bar{u}^m}{\partial u^r} \right) \left(\frac{\partial u^s}{\partial \bar{u}^i} \frac{\partial u^t}{\partial \bar{u}^j} \right) T_{st}^{pqr} \\ &= \left(\bar{a}_p^k \bar{a}_q^l \bar{a}_r^m \right) \left(\underline{a}_i^s \underline{a}_j^t \right) T_{st}^{pqr} \end{aligned} \tag{2.144}$$

2.5 Tensor Calculus in General Curvilinear Coordinates

In the following sections, some necessary symbols, such as the Christoffel symbols, Riemann–Christoffel tensor, and fundamental invariants of the Nabla operator have to be taken into account in the tensor applications to fluid mechanics and other working areas.

2.5.1 Physical Component of Tensors

Various types of the second-order tensors are shown in Eq. (2.74). The physical tensor component is defined as the tensor component on its covariant unitary basis

$\mathbf{g}_i^*$. Therefore, the basis of the general curvilinear coordinates must be normalized (cf. Appendix B).

Dividing the covariant basis by its vector length, the covariant unitary basis (covariant normalized basis) results in

$$\mathbf{g}_i^* = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} = \frac{\mathbf{g}_i}{\sqrt{g_{(\mathrm{ii})}}} \Rightarrow |\mathbf{g}_i^*| = 1 \tag{2.145}$$

The covariant basis norm $|\mathbf{g}_i|$ can be considered as a scale factor h_i.

$$h_i = |\mathbf{g}_i| = \sqrt{g_{(\mathrm{ii})}} \quad (\text{no summation over i})$$

Thus, the covariant basis can be related to its covariant unitary basis by the relation of

$$\mathbf{g}_i = \sqrt{g_{(\mathrm{ii})}}\mathbf{g}_i^* = h_i \mathbf{g}_i^* \tag{2.146}$$

The contravariant basis can be related to its covariant unitary basis using Eqs. (2.47) and (2.146).

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j = g^{ij} h_j \mathbf{g}_j^* \tag{2.147}$$

The contravariant second-order tensor can be written in the covariant unitary bases using Eq. (2.146).

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j = \left(T^{ij}h_i h_j\right)\mathbf{g}_i^*\mathbf{g}_j^* \equiv T^{*ij}\mathbf{g}_i^*\mathbf{g}_j^* \tag{2.148}$$

Thus, the physical contravariant tensor components result in

$$T^{*ij} \equiv T^{ij} h_i h_j \tag{2.149}$$

The covariant second-order tensor can be written in the contravariant unitary bases using Eq. (2.147).

$$\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j = \left(T_{ij}g^{ik}g^{jl}h_k h_l\right)\mathbf{g}_k^*\mathbf{g}_l^* \equiv T_{ij}^*\mathbf{g}_k^*\mathbf{g}_l^* \tag{2.150}$$

Similarly, the physical covariant tensor component results in

$$T_{ij}^* = T_{ij}g^{ik}g^{jl}h_k h_l \tag{2.151}$$

The mixed tensors can be written in the covariant unitary bases using Eqs. (2.146) and (2.147)

$$\begin{aligned}\mathbf{T} &= T_j^i\mathbf{g}_i\mathbf{g}^j = T_j^i\mathbf{g}_i\left(g^{jk}\mathbf{g}_k\right) \\ &= T_j^i\left(h_i\mathbf{g}_i^*\right)\left(g^{jk}h_k\mathbf{g}_k^*\right) = \left(T_j^i g^{jk}h_i h_k\right)\mathbf{g}_i^*\mathbf{g}_k^* \\ &\equiv \left(T_j^i\right)^*\mathbf{g}_i^*\mathbf{g}_k^*\end{aligned} \tag{2.152}$$

Thus, the physical mixed tensor component results from Eq. (2.152):

$$\left(T_j^i\right)^* = T_j^i g^{jk} h_i h_k \tag{2.153}$$

2.5.2 Derivatives of Covariant Bases

Let $\mathbf{g}_i$ be a covariant basis in the curvilinear coordinates $\{u^i\}$, the derivative of the covariant basis with respect to the time variable t can be computed as

$$\dot{\mathbf{g}}_i = \frac{\partial \mathbf{g}_i}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{r}}{\partial u^i}\right) \equiv \dot{\mathbf{r}}_{,i} \tag{2.154}$$

Due to u^i is a differentiable function of t, Eq. (2.154) can be rewritten as

$$\dot{\mathbf{g}}_i = \frac{\partial \mathbf{g}_i}{\partial t} = \frac{\partial \mathbf{g}_i}{\partial u^j}\frac{\partial u^j}{\partial t} \equiv \mathbf{g}_{i,j}\dot{u}^j \tag{2.155}$$

where $\mathbf{g}_{i,j}$ is called the derivative of the covariant basis $\mathbf{g}_i$ of the curvilinear coordinates $\{u^i\}$.

Using the chain rule of differentiation, the covariant basis of the curvilinear coordinates $\{u^i\}$ can be calculated in Cartesian coordinates $\{x^i\}$.

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial \mathbf{r}}{\partial x^p}\frac{\partial x^p}{\partial u^i} = \mathbf{e}_p x^p_{,i} \tag{2.156}$$

Similarly, one obtains the covariant basis of the coordinates $\{x^i\}$.

$$\mathbf{e}_p = \frac{\partial \mathbf{r}}{\partial x^p} = \frac{\partial \mathbf{r}}{\partial u^k}\frac{\partial u^k}{\partial x^p} = \mathbf{g}_k u^k_{,p} \tag{2.157}$$

The derivative of the covariant basis of the coordinates $\{u^i\}$ can be obtained from Eqs. (2.156) and (2.157).

$$\begin{aligned}
\mathbf{g}_{i,j} &= \frac{\partial \mathbf{g}_i}{\partial u^j} = \frac{\partial\left(\mathbf{e}_p x^p_{,i}\right)}{\partial u^j} = \mathbf{e}_p \frac{\partial x^p_{,i}}{\partial u^j} \\
&= u^k_{,p}\frac{\partial x^p_{,i}}{\partial u^j}\mathbf{g}_k = \left(\frac{\partial u^k}{\partial x^p}\frac{\partial^2 x^p}{\partial u^i \partial u^j}\right)\mathbf{g}_k \\
&\equiv \Gamma^k_{ij}\mathbf{g}_k \quad \text{for } k = 1, 2, \ldots, N
\end{aligned} \tag{2.158}$$

The symbol Γ^k_{ij} in Eq. (2.158) is defined as the second-kind Christoffel symbol, which has 27 (= 3^3) components for a three-dimensional space ($N = 3$).

Thus, the second-order Christoffel symbols that only depend on both coordinates of $\{u^i\}$ and $\{x^i\}$ can be written as

$$\begin{aligned}\Gamma^k_{ij} &= \frac{\partial u^k}{\partial x^p}\frac{\partial^2 x^p}{\partial u^i \partial u^j}\\ &= \frac{\partial u^k}{\partial x^p}\frac{\partial^2 x^p}{\partial u^j \partial u^i} = \Gamma^k_{ji}\end{aligned} \tag{2.159}$$

The result of Eq. (2.159) proves that the second-kind Christoffel symbols are symmetric with respect to i and j.

The second-kind Christoffel symbols are given by multiplying both sides of Eq. (2.158) by the contravariant basis $\mathbf{g}^l$.

$$\begin{aligned}\Gamma^k_{ij}\left(\mathbf{g}_k.\mathbf{g}^l\right) = \Gamma^k_{ij}\delta^l_k &= \mathbf{g}^l.\mathbf{g}_{i,j}\\ \Rightarrow \Gamma^l_{ij} &= \mathbf{g}^l.\mathbf{g}_{i,j}\end{aligned} \tag{2.160}$$

Substituting Eq. (2.158) into Eq. (2.155), one obtains the relation between the covariant basis time derivative and the Christoffel symbol.

$$\dot{\mathbf{g}}_i = \mathbf{g}_{i,j}\dot{u}^j = \Gamma^k_{ij}\dot{u}^j\mathbf{g}_k \tag{2.161}$$

Furthermore, the covariant basis derivative can be calculated in Cartesian coordinate $\{x^i\}$ using Eq. (2.156).

$$\mathbf{g}_{i,j} = \frac{\partial \mathbf{g}_i}{\partial u^j} = \frac{\partial\left(\mathbf{e}_p x^p_{,i}\right)}{\partial u^j} = \mathbf{e}_p\frac{\partial\left(x^p_{,i}\right)}{\partial u^j} = \mathbf{e}_p x^p_{,ij} \tag{2.162}$$

According to Eq. (2.159), the second-kind Christoffel symbols can be rewritten as

$$\begin{aligned}\Gamma^k_{ij} &= \frac{\partial u^k}{\partial x^p}\frac{\partial^2 x^p}{\partial u^i \partial u^j} = \frac{\partial u^k}{\partial x^p}\frac{\partial^2 x^p}{\partial u^j \partial u^i}\\ &= u^k_{,p}x^p_{,ij} = u^k_{,p}x^p_{,ji}\end{aligned} \tag{2.163}$$

2.5.3 Christoffel Symbols of First and Second Kind

According to Eq. (2.160), the second-kind Christoffel symbol can be defined as

$$\begin{aligned}\Gamma_{ij}^{k} &\equiv \left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} = \mathbf{g}^{k} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^{i} \partial u^{j}} = \mathbf{g}^{k} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^{j} \partial u^{i}} \\ &= \mathbf{g}^{k} \cdot \mathbf{g}_{i,j} = \mathbf{g}^{k} \cdot \mathbf{g}_{j,i} = \Gamma_{ji}^{k}\end{aligned} \tag{2.164}$$

Equation (2.164) reconfirms the symmetric property of the Christoffel symbols with respect to the indices i and j. Obviously, the Christoffel symbols are coordinate dependent; therefore, they are not tensors.

In order to compute the second-kind Christoffel symbols in the covariant metric coefficients, the derivative of g_{ij} with respect to u^k has to be taken into account.

$$\begin{aligned} g_{ij} &= \left(\mathbf{g}_i \cdot \mathbf{g}_j\right) \\ \Rightarrow g_{ij,k} &\equiv \frac{\partial g_{ij}}{\partial u^k} = \left(\mathbf{g}_i \cdot \mathbf{g}_j\right)_{,k} = \mathbf{g}_{i,k} \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \mathbf{g}_{j,k}\end{aligned} \tag{2.165}$$

Using Eq. (2.158) at changing the index j into k; then, i into j, one obtains the following relations

$$\mathbf{g}_{i,k} = \Gamma_{ik}^{p} \mathbf{g}_p ; \mathbf{g}_{j,k} = \Gamma_{jk}^{p} \mathbf{g}_p \tag{2.166}$$

Substituting Eq. (2.166) into Eq. (2.165), one obtains the derivative of g_{ij} with respect to u^k.

$$\begin{aligned} g_{ij,k} &= \mathbf{g}_{i,k} \cdot \mathbf{g}_j + \mathbf{g}_{j,k} \cdot \mathbf{g}_i \\ &= \Gamma_{ik}^{p} \mathbf{g}_p \cdot \mathbf{g}_j + \Gamma_{jk}^{p} \mathbf{g}_p \cdot \mathbf{g}_i \\ &= \Gamma_{ik}^{p} g_{pj} + \Gamma_{jk}^{p} g_{p\,i}\end{aligned} \tag{2.167}$$

Interchanging k with i in Eq. (2.167), one obtains

$$g_{kj,i} = \Gamma_{ki}^{p} g_{pj} + \Gamma_{ji}^{p} g_{p\,k} \tag{2.168}$$

Analogously, one reaches the relation interchanging k with j in Eq. (2.167).

$$g_{ik,j} = \Gamma_{ij}^{p} g_{pk} + \Gamma_{kj}^{p} g_{p\,i} \tag{2.169}$$

Combining Eqs. (2.167), (2.168), and (2.169), the Christoffel symbols can be written in the derivatives of the covariant metric coefficients.

$$g_{pj} \Gamma_{ik}^{p} = \frac{1}{2}\left(g_{ij,k} + g_{kj,i} - g_{ik,j}\right) \tag{2.170}$$

Multiplying Eq. (2.170) by g^{qj}, the Christoffel symbols result according to Eq. (2.50) in

$$\begin{aligned} \Gamma^p_{ik} g_{pj} g^{qj} &= \Gamma^p_{ik} \delta^q_p \\ &\Rightarrow \Gamma^q_{ik} = \frac{1}{2} g^{qj} \left(g_{ij,k} + g_{kj,i} - g_{ik,j} \right) \end{aligned} \tag{2.171}$$

Changing j into p, k into j, and q into k in Eq. (2.171), one obtains

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2} \left(g_{ip,j} + g_{jp,i} - g_{ij,p} \right) g^{kp} \\ &\equiv \Gamma_{ijp} g^{kp} \end{aligned} \tag{2.172}$$

Changing the index p into k, the first-kind Christoffel symbol Γ_{ijp} in Eq. (2.172) that has 27 (= 3^3) components for a three-dimensional space ($N = 3$) is defined as

$$\begin{aligned} \Gamma_{ijk} &\equiv [ij, k] \\ &\equiv \frac{1}{2} \left(g_{ik,j} + g_{jk,i} - g_{ij,k} \right) \\ &= g_{pk} \Gamma^p_{ij} \quad \text{for} \quad p = 1, 2, \ldots, N \end{aligned} \tag{2.173a}$$

Other expressions of the Christoffel symbols can be found in some literature.

$$\begin{aligned} \Gamma_{ijk} &\equiv [i\, j, k] \\ &= g_{pk} \left\{ \begin{matrix} p \\ i \quad j \end{matrix} \right\} \equiv g_{pk} \Gamma^p_{ij} \quad \text{for} \quad p = 1, 2, \ldots, N \end{aligned} \tag{2.173b}$$

2.5.4 Prove that the Christoffel Symbols are Symmetric

1. The first-kind Christoffel symbol is symmetric with respect to i, j

   According to Eq. (2.173a), the first-kind Christoffel symbol can be written as

$$\Gamma_{ijk} = \frac{1}{2} \left(g_{ik,j} + g_{jk,i} - g_{ij,k} \right)$$

   Interchanging i with j in the terms on the RHS, one obtains

$$\begin{aligned} \Gamma_{ijk} &= \frac{1}{2} \left(g_{jk,i} + g_{ik,j} - g_{ji,k} \right) \\ &= \Gamma_{jik} (qed.) \end{aligned}$$

2. The second-kind Christoffel symbol is symmetric with respect to i, j

   Using Eq. (2.172), the second-kind Christoffel symbol can be expressed as

$$\Gamma_{ij}^{k} = g^{kp}\,\Gamma_{ijp}$$

Due to the symmetry of the first-kind Christoffel symbol, the second-kind Christoffel symbol results in

$$\begin{aligned}\Gamma_{ij}^{k} &= g^{kp}\,\Gamma_{ijp} = g^{kp}\,\Gamma_{jip} \\ &= \Gamma_{ji}^{k}\,(qed)\end{aligned}$$

2.5.5 Examples of Computing the Christoffel Symbols

Given a curvilinear coordinate$\{u^i\}$ with $u^1 = u;\ u^2 = v;\ u^3 = w$ in another coordinate $\{x^i\}$, the relation between two coordinate systems can be written as

$$\begin{cases} x^1 = uv \\ x^2 = w \\ x^3 = u^2 - v \end{cases}$$

The covariant basis matrix **G** can be calculated from

$$\mathbf{G} = [\mathbf{g}_1 \quad \mathbf{g}_2 \quad \mathbf{g}_3] = \begin{pmatrix} \dfrac{\partial x^1}{\partial u^1} & \dfrac{\partial x^1}{\partial u^2} & \dfrac{\partial x^1}{\partial u^3} \\ \dfrac{\partial x^2}{\partial u^1} & \dfrac{\partial x^2}{\partial u^2} & \dfrac{\partial x^2}{\partial u^3} \\ \dfrac{\partial x^3}{\partial u^1} & \dfrac{\partial x^3}{\partial u^2} & \dfrac{\partial x^3}{\partial u^3} \end{pmatrix} = \begin{bmatrix} v & u & 0 \\ 0 & 0 & 1 \\ 2u & -1 & 0 \end{bmatrix}$$

Therefore, the covariant bases can be given in

$$\begin{cases} \mathbf{g}_1 = (v,\ 0,\ 2u) \\ \mathbf{g}_2 = (u,\ 0,\ -1) \\ \mathbf{g}_3 = (0,\ 1,\ 0) \end{cases}$$

The determinant of **G** that equals the Jacobian J of

$$|\mathbf{G}| = J = \begin{vmatrix} v & u & 0 \\ 0 & 0 & 1 \\ 2u & -1 & 0 \end{vmatrix} = 2u^2 + v \neq 0$$

The contravariant basis matrix $\mathbf{G}^{-1}$ is the inverse matrix of the covariant basis matrix **G**.

$$\mathbf{G}^{-1} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{bmatrix} = \begin{pmatrix} \dfrac{\partial u^1}{\partial x^1} & \dfrac{\partial u^1}{\partial x^2} & \dfrac{\partial u^1}{\partial x^3} \\ \dfrac{\partial u^2}{\partial x^1} & \dfrac{\partial u^2}{\partial x^2} & \dfrac{\partial u^2}{\partial x^3} \\ \dfrac{\partial u^3}{\partial x^1} & \dfrac{\partial u^3}{\partial x^2} & \dfrac{\partial u^3}{\partial x^3} \end{pmatrix}$$

$$= \frac{1}{J}\begin{bmatrix} 1 & 0 & u \\ 2u & 0 & -v \\ 0 & (2u^2+v) & 0 \end{bmatrix} = \begin{bmatrix} J^{-1} & 0 & uJ^{-1} \\ 2uJ^{-1} & 0 & -vJ^{-1} \\ 0 & 1 & 0 \end{bmatrix}$$

Thus,

$$\left|\mathbf{G}^{-1}\right| = \frac{1}{(2u^2+v)} = \frac{1}{J}$$

$$A_{ij}^k B_k^{il} = C_j^l \Rightarrow \bar{A}_{ij}^k \bar{B}_k^{il} = \bar{C}_j^l$$

Thus, the contravariant bases result in

$$\begin{cases} \mathbf{g}^1 = J^{-1}(1,\ 0,\ u) \\ \mathbf{g}^2 = J^{-1}(2u,\ 0,\ -v) \\ \mathbf{g}^3 = J^{-1}(0,\ 2u^2+v,\ 0) \end{cases}$$

Some examples of the second-kind Christoffel symbols of 27 components can be computed from Eq. (2.160).

$$\begin{aligned} \Gamma_{ij}^k &= \mathbf{g}_{i,j}\,\mathbf{g}^k \Rightarrow \\ \Gamma_{11}^1 &= \mathbf{g}_{1,1}\,\mathbf{g}^1 = J^{-1}(0.1+0.0+2.u) = 2uJ^{-1} \\ \Gamma_{12}^1 &= \mathbf{g}_{1,2}\,\mathbf{g}^1 = J^{-1}(1.1+0.0+0.u) = J^{-1} \\ \Gamma_{13}^1 &= \mathbf{g}_{1,3}\,\mathbf{g}^1 = J^{-1}(0.1+0.0+0.u) = 0 \\ &\ldots \\ \Gamma_{32}^3 &= \mathbf{g}_{3,2}\,\mathbf{g}^3 = J^{-1}(0.0+0.(2u^2+v)+0.0) = 0 \\ \Gamma_{33}^3 &= \mathbf{g}_{3,3}\,\mathbf{g}^3 = J^{-1}(0.0+0.(2u^2+v)+0.0) = 0 \end{aligned}$$

The first-kind Christoffel symbols containing 27 components in $\mathbf{R}^3$ can be computed from Eq. (2.173a).

$$\Gamma_{ijk} = g_{pk}\Gamma_{ij}^p = (\mathbf{g}_p.\mathbf{g}_k)\Gamma_{ij}^p \quad \text{for} \quad p = 1,\ 2,\ 3$$

2.5.6 Coordinate Transformations of the Christoffel Symbols

The second-kind Christoffel symbols like tensor components strongly depend on the coordinates at the coordinate transformations. The curvilinear coordinates $\{u^i\}$ is transformed into the new barred curvilinear coordinates $\{\bar{u}^i\}$. Therefore, the old basis is also changed into the new basis.

The second-kind Christoffel symbols can be written in the new basis of the barred coordinates $\{\bar{u}^i\}$.

$$\bar{\Gamma}_{ij}^k = \bar{\mathbf{g}}^k \cdot \frac{\partial \bar{\mathbf{g}}_i}{\partial u^j} = \bar{\mathbf{g}}^k . \bar{\mathbf{g}}_{i,j} \tag{2.174}$$

Using the chain rule of differentiation, the basis of the coordinates $\{u^i\}$ can be calculated as

$$\begin{aligned} \mathbf{g}_p &= \frac{\partial \mathbf{r}}{\partial u^p} = \frac{\partial \mathbf{r}}{\partial \bar{u}^k} \frac{\partial \bar{u}^k}{\partial u^p} \\ &= \bar{\mathbf{g}}_k \frac{\partial \bar{u}^k}{\partial u^p} \end{aligned} \tag{2.175}$$

Multiplied Eq. (2.175) by the new contravariant basis of the coordinates $\{\bar{u}^i\}$, one obtains changing the indices m into k, and p into l.

$$\begin{aligned} \bar{\mathbf{g}}^m . \mathbf{g}_p &= \bar{\mathbf{g}}^m . \bar{\mathbf{g}}_k \frac{\partial \bar{u}^k}{\partial u^p} = \delta_k^m \frac{\partial \bar{u}^k}{\partial u^p} = \frac{\partial \bar{u}^m}{\partial u^p} \\ &\Rightarrow (\bar{\mathbf{g}}^m . \mathbf{g}_p) \mathbf{g}^p = \bar{\mathbf{g}}^m (\mathbf{g}_p . \mathbf{g}^p) = \bar{\mathbf{g}}^m \delta_p^p = \frac{\partial \bar{u}^m}{\partial u^p} \mathbf{g}^p \\ &\Rightarrow \bar{\mathbf{g}}^k = \frac{\partial \bar{u}^k}{\partial u^l} \mathbf{g}^l \end{aligned} \tag{2.176}$$

The new covariant basis of the coordinates $\{\bar{u}^i\}$ can be calculated as

$$\bar{\mathbf{g}}_i = \frac{\partial \mathbf{r}}{\partial \bar{u}^i} = \frac{\partial \mathbf{r}}{\partial u^p} \frac{\partial u^p}{\partial \bar{u}^i} = \mathbf{g}_p \frac{\partial u^p}{\partial \bar{u}^i} \tag{2.177}$$

Thus, the new covariant basis derivative with respect to j of the coordinates $\{\bar{u}^i\}$ results in

$$\begin{aligned} \bar{\mathbf{g}}_{i,j} &= \frac{\partial \bar{\mathbf{g}}_i}{\partial \bar{u}^j} = \frac{\partial}{\partial \bar{u}^j} \left(\frac{\partial u^p}{\partial \bar{u}^i} \mathbf{g}_p \right) \\ &= \frac{\partial^2 u^p}{\partial \bar{u}^j \partial \bar{u}^i} \mathbf{g}_p + \frac{\partial u^p}{\partial \bar{u}^i} \frac{\partial \mathbf{g}_p}{\partial \bar{u}^j} \end{aligned} \tag{2.178}$$

Substituting Eqs. (2.176) and (2.178) into Eq. (2.174), one obtains the second-kind Christoffel symbols in the new basis.

$$\begin{aligned}\bar{\Gamma}^k_{ij} &= \bar{\mathbf{g}}^k.\bar{\mathbf{g}}_{i,j}\\ &= \frac{\partial \bar{u}^k}{\partial u^l}\mathbf{g}^l.\left(\frac{\partial^2 u^p}{\partial \bar{u}^j \partial \bar{u}^i}\mathbf{g}_p + \frac{\partial u^p}{\partial \bar{u}^i}\frac{\partial \mathbf{g}_p}{\partial \bar{u}^j}\right)\end{aligned} \tag{2.179}$$

Using Eq. (2.166) and the chain rule of differentiation, the second term in the parentheses of Eq. (2.179) can be computed as

$$\begin{aligned}\frac{\partial \mathbf{g}_p}{\partial \bar{u}^j} &= \frac{\partial \mathbf{g}_p}{\partial u^q}\frac{\partial u^q}{\partial \bar{u}^j}\\ &= \frac{\partial u^q}{\partial \bar{u}^j}\mathbf{g}_{p,q} = \frac{\partial u^q}{\partial \bar{u}^j}\left(\Gamma^r_{pq}\mathbf{g}_r\right)\end{aligned} \tag{2.180}$$

Inserting Eq. (2.180) into Eq. (2.179), the transformed Christoffel symbols in the new barred coordinates can be calculated as

$$\begin{aligned}\bar{\Gamma}^k_{ij} &= \bar{\mathbf{g}}^k.\bar{\mathbf{g}}_{i,j}\\ &= \frac{\partial \bar{u}^k}{\partial u^l}\left(\frac{\partial^2 u^p}{\partial \bar{u}^j \partial \bar{u}^i}(\mathbf{g}^l.\mathbf{g}_p) + \frac{\partial u^p}{\partial \bar{u}^i}\frac{\partial u^q}{\partial \bar{u}^j}\Gamma^r_{pq}(\mathbf{g}^l.\mathbf{g}_r)\right)\\ &= \frac{\partial \bar{u}^k}{\partial u^l}\left(\frac{\partial^2 u^p}{\partial \bar{u}^j \partial \bar{u}^i}\delta^l_p + \frac{\partial u^p}{\partial \bar{u}^i}\frac{\partial u^q}{\partial \bar{u}^j}\Gamma^r_{pq}\delta^l_r\right)\\ &= \frac{\partial \bar{u}^k}{\partial u^l}\left(\frac{\partial^2 u^l}{\partial \bar{u}^i \partial \bar{u}^j} + \frac{\partial u^p}{\partial \bar{u}^i}\frac{\partial u^q}{\partial \bar{u}^j}\Gamma^l_{pq}\right)\end{aligned} \tag{2.181}$$

Therefore, the transformed Christoffel symbols in the new barred coordinates $\{\bar{u}^{\mathrm{i}}\}$ result in

$$\bar{\Gamma}^k_{ij} = \Gamma^l_{pq}\frac{\partial \bar{u}^k}{\partial u^l}\frac{\partial u^p}{\partial \bar{u}^i}\frac{\partial u^q}{\partial \bar{u}^j} + \frac{\partial \bar{u}^k}{\partial u^l}\frac{\partial^2 u^l}{\partial \bar{u}^i \partial \bar{u}^j} \tag{2.182}$$

Rearranging the terms in Eq. (2.181), the second derivatives of u^l with respect to the new barred coordinates result in

$$\begin{aligned}\frac{\partial u^l}{\partial \bar{u}^k}\bar{\Gamma}^k_{ij} &= \left(\frac{\partial^2 u^l}{\partial \bar{u}^i \partial \bar{u}^j} + \frac{\partial u^p}{\partial \bar{u}^i}\frac{\partial u^q}{\partial \bar{u}^j}\Gamma^l_{pq}\right)\\ \Rightarrow \frac{\partial^2 u^l}{\partial \bar{u}^i \partial \bar{u}^j} &= \frac{\partial u^l}{\partial \bar{u}^k}\bar{\Gamma}^k_{ij} - \frac{\partial u^p}{\partial \bar{u}^i}\frac{\partial u^q}{\partial \bar{u}^j}\Gamma^l_{pq}\end{aligned} \tag{2.183}$$

Using Eq. (2.160), all Christoffel symbols in Cartesian coordinates $\{x^i\}$ vanish because the basis $\mathbf{e}_i$ does not change in any coordinate x^j.

$$\Gamma^k_{ij} = \mathbf{e}^k . \mathbf{e}_{i,j} = \mathbf{e}^k . \frac{\partial \mathbf{e}_i}{\partial x^j} = 0 \tag{2.184}$$

2.5.7 Derivatives of Contravariant Bases

Like Eq. (2.158), the derivative of the contravariant basis of the curvilinear coordinates $\{u^i\}$ with respect to u^j can be defined as

$$\mathbf{g}^i_{,j} = \frac{\partial \mathbf{g}^i}{\partial u^j} \equiv \hat{\Gamma}^i_{jk} \mathbf{g}^k \tag{2.185}$$

where $\hat{\Gamma}^i_{jk}$ are the second-kind Christoffel symbols in the contravariant bases $\mathbf{g}^k$.

In order to compute those Christoffel symbols, some calculating steps are carried out in the following section.

The derivative of the product between the covariant and contravariant bases with respect to u^j can be computed using Eqs. (2.156), (2.164), and (2.185).

$$\begin{aligned} \left(\mathbf{g}^i . \mathbf{g}_j\right)_{,k} &= \mathbf{g}^i_{,k} . \mathbf{g}_j + \mathbf{g}^i . \mathbf{g}_{j,k} = \left(\delta^i_j\right)_{,k} \\ &= \hat{\Gamma}^i_{kl} \left(\mathbf{g}^l . \mathbf{g}_{\mathrm{j}}\right) + \Gamma^l_{jk} \left(\mathbf{g}_l . \mathbf{g}^i\right) \\ &= \hat{\Gamma}^i_{kl} \delta^l_j + \Gamma^l_{jk} \delta^i_l \\ &= \hat{\Gamma}^i_{kj} + \Gamma^i_{jk} = \hat{\Gamma}^i_{kj} + \Gamma^i_{kj} \\ &= 0 \end{aligned} \tag{2.186}$$

Thus, the relation between the Christoffel symbols of two coordinates results in

$$\hat{\Gamma}^i_{kj} = -\Gamma^i_{kj} = -\Gamma^i_{jk} \tag{2.187}$$

Using Eqs. (2.164) and (2.187), one obtains

$$\hat{\Gamma}^i_{kj} = -\Gamma^i_{kj} = -\Gamma^i_{jk} = \hat{\Gamma}^i_{jk} \tag{2.188}$$

It proves that the Christoffel symbol $\hat{\Gamma}^i_{jk}$ is symmetric with respect to j and k.

Finally, the derivatives of the contravariant basis $\mathbf{g}^i$ with respect to u^j result from Eqs. (2.185) and (2.187).

$$\begin{aligned} \mathbf{g}^i_{,j} &= \hat{\Gamma}^i_{jk} \mathbf{g}^k = -\Gamma^i_{jk} \mathbf{g}^k \\ &= \hat{\Gamma}^i_{kj} \mathbf{g}^k = -\Gamma^i_{kj} \mathbf{g}^k \end{aligned} \tag{2.189}$$

2.5.8 Derivatives of Covariant Metric Coefficients

The derivatives of the covariant metric coefficient can be derived from the first-kind Christoffel symbols written as

$$\begin{aligned} \Gamma_{ikj} &= \frac{1}{2}\left(g_{ij,k} + g_{kj,i} - g_{ik,j}\right); \\ \Gamma_{jki} &= \frac{1}{2}\left(g_{ji,k} + g_{ki,j} - g_{jk,i}\right) \end{aligned} \tag{2.190}$$

The derivative of the covariant metric coefficient results by adding both Christoffel symbols given in Eq. (2.190).

$$\begin{aligned} \Gamma_{ikj} + \Gamma_{jki} &= \frac{1}{2}(g_{ij,k} + g_{kj,i} - g_{ik,j}) + \frac{1}{2}(g_{ji,k} + g_{ki,j} - g_{jk,i}) \\ &= \frac{1}{2}(g_{ij,k} + \underset{\cdots\cdots}{g_{kj,i}} - \underline{g_{ik,j}}\ ) + \frac{1}{2}(g_{ij,k} + \underline{g_{ik,j}} - \underset{\cdots\cdots}{g_{kj,i}}) \\ &= g_{ij,k} \end{aligned} \tag{2.191}$$

Therefore, the derivatives of the covariant metric coefficient g_{ij} with respect to u^k can be expressed in the first-kind Christoffel symbols.

$$g_{ij,k} \equiv \frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ikj} + \Gamma_{jki} \tag{2.192}$$

Using Eq. (2.173a), Eq. (2.192) can be rewritten in the second-kind Christoffel symbols.

$$\begin{aligned} g_{ij,k} &= \Gamma_{ikj} + \Gamma_{jki} \\ &= g_{jl}\Gamma_{ik}^{l} + g_{il}\Gamma_{jk}^{l} \end{aligned} \tag{2.193}$$

Similar to Eq. (2.192), one can write

$$g_{jk,i} = \Gamma_{jik} + \Gamma_{kij} \tag{2.194}$$

Subtracting Eq. (2.192) from Eq. (2.194), one obtains the relation

$$\begin{aligned} g_{ij,k} - g_{jk,i} &= \underline{\Gamma_{ikj}} + \Gamma_{jki} - \Gamma_{jik} - \underline{\Gamma_{kij}} \\ &= \Gamma_{jki} - \Gamma_{jik} \\ &= \Gamma_{kji} - \Gamma_{ijk} \end{aligned} \tag{2.195}$$

2.5.9 *Covariant Derivatives of Tensors*

(a) **Contravariant first-order tensors with components T^i**

The contravariant first-order tensor (vector) **T** can be written in the covariant basis.

$$\mathbf{T} = T^i \mathbf{g}_i \tag{2.196}$$

Using Eq. (2.158), the derivative of the contravariant tensor **T** with respect to u^j results in

$$\begin{aligned}\mathbf{T}_{,j} &= \left(T^i \mathbf{g}_i\right)_{,j} = T^i_{,j}\mathbf{g}_i + T^i \mathbf{g}_{i,j} \\ &= T^i_{,j}\mathbf{g}_i + T^i \Gamma^k_{ij}\mathbf{g}_k\end{aligned} \tag{2.197}$$

The derivative of the contravariant tensor component T^i with respect to u^j in Eq. (2.197) can be defined as

$$T^i_{,j} \equiv \frac{\partial T^i}{\partial u^j} \tag{2.198}$$

Interchanging i with k in the second term in the RHS of Eq. (2.197), one obtains

$$\begin{aligned}T^i \Gamma^k_{ij}\mathbf{g}_k &= T^k \Gamma^i_{kj}\mathbf{g}_i \\ &= T^k \Gamma^i_{jk}\mathbf{g}_i\end{aligned} \tag{2.199}$$

Substituting Eq. (2.199) into Eq. (2.197), one obtains the derivative of **T** with respect to u^j.

$$\begin{aligned}\mathbf{T}_{,j} &= T^i_{,j}\mathbf{g}_i + T^k \Gamma^i_{jk}\mathbf{g}_i \\ &= (T^i_{,j} + \Gamma^i_{jk}T^k)\mathbf{g}_i \\ &\equiv T^i\big|_j \mathbf{g}_i\end{aligned} \tag{2.200}$$

Therefore, the covariant derivative with respect to u^j of the contravariant first-order tensor (vector) can be written as

$$T^i\big|_j = T^i_{,j} + \Gamma^i_{jk}T^k = \mathbf{T}_{,j}.\mathbf{g}^i \tag{2.201}$$

The covariant derivative of the contravariant first-order tensor component is transformed in the new barred coordinates $\{\bar{u}^i\}$.

$$\begin{aligned}T^i\big|_k &= \bar{T}^j\big|_n \frac{\partial u^i}{\partial \bar{u}^j}\frac{\partial \bar{u}^n}{\partial u^k} \\ \Rightarrow \bar{T}^j\big|_n &= T^i\big|_k \frac{\partial \bar{u}^j}{\partial u^i}\frac{\partial u^k}{\partial \bar{u}^n}\end{aligned} \tag{2.202a}$$

Proof The first-order tensor component can be written as

$$T^i = \bar{T}^j \frac{\partial u^i}{\partial \bar{u}^j} \tag{2.202b}$$

Differentiating T^i with respect to u^k and using the chain rule of differentiation, one obtains

$$\begin{aligned} \frac{\partial T^i}{\partial u^k} &= \left(\bar{T}^j \frac{\partial u^i}{\partial \bar{u}^j} \right)_{,k} \\ &= \left(\frac{\partial \bar{T}^j}{\partial \bar{u}^n} \frac{\partial \bar{u}^n}{\partial u^k} \right) \frac{\partial u^i}{\partial \bar{u}^j} + \bar{T}^j \left(\frac{\partial^2 u^i}{\partial \bar{u}^j \partial \bar{u}^n} \right) \frac{\partial \bar{u}^n}{\partial u^k} \end{aligned} \tag{2.202c}$$

Using Eq. (2.183), we have

$$\frac{\partial^2 u^i}{\partial \bar{u}^n \partial \bar{u}^j} = \bar{\Gamma}^k_{nj} \frac{\partial u^i}{\partial \bar{u}^k} - \Gamma^i_{pq} \frac{\partial u^p}{\partial \bar{u}^n} \frac{\partial u^q}{\partial \bar{u}^j} \tag{2.202d}$$

Substituting Eq. (2.202d) into Eq. (2.202c) and interchanging the indices k with j and j with m, one obtains

$$\frac{\partial T^i}{\partial u^k} = \left(\frac{\partial \bar{T}^j}{\partial \bar{u}^n} \frac{\partial \bar{u}^n}{\partial u^k} \right) \frac{\partial u^i}{\partial \bar{u}^j} + \bar{T}^j \left(\bar{\Gamma}^k_{nj} \frac{\partial u^i}{\partial \bar{u}^k} - \Gamma^i_{pq} \frac{\partial u^p}{\partial \bar{u}^n} \frac{\partial u^q}{\partial \bar{u}^j} \right) \frac{\partial \bar{u}^n}{\partial u^k}$$

thus,

$$\begin{aligned} \frac{\partial T^i}{\partial u^k} + \bar{T}^j \Gamma^i_{pq} \left(\frac{\partial u^p}{\partial \bar{u}^n} \frac{\partial u^q}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \right) &= \frac{\partial \bar{T}^j}{\partial \bar{u}^n} \frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} + \bar{T}^j \bar{\Gamma}^k_{nj} \frac{\partial u^i}{\partial \bar{u}^k} \frac{\partial \bar{u}^n}{\partial u^k} \\ &= \frac{\partial \bar{T}^j}{\partial \bar{u}^n} \left(\frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \right) + \bar{T}^m \bar{\Gamma}^j_{nm} \left(\frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \right) \end{aligned} \tag{2.202e}$$

The terms in the RHS of Eq. (2.202e) can be written as

$$\begin{aligned} \frac{\partial \bar{T}^j}{\partial \bar{u}^n} \left(\frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \right) + \bar{T}^m \bar{\Gamma}^j_{nm} \left(\frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \right) &= \left[\bar{T}^j_{,n} + \bar{T}^m \bar{\Gamma}^j_{nm} \right] \frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \\ &= \bar{T}^j \big|_n \frac{\partial u^i}{\partial \bar{u}^j} \frac{\partial \bar{u}^n}{\partial u^k} \end{aligned} \tag{2.202f}$$

Using Eq. (2.202b) and interchanging the indices p with k and q with m, the terms in the LHS of Eq. (2.202e) are rearranged in

$$\begin{aligned}\frac{\partial T^i}{\partial u^k} + \bar{T}^j \Gamma^i_{pq}\left(\frac{\partial u^p}{\partial \bar{u}^n}\frac{\partial u^q}{\partial \bar{u}^j}\frac{\partial \bar{u}^n}{\partial u^k}\right) &= T^i_{,k} + \left(T^m \frac{\partial \bar{u}^j}{\partial u^m}\right)\Gamma^i_{pq}\left(\frac{\partial u^p}{\partial \bar{u}^n}\frac{\partial u^q}{\partial \bar{u}^j}\frac{\partial \bar{u}^n}{\partial u^k}\right) \\ &= T^i_{,k} + T^m \Gamma^i_{km}\left(\frac{\partial u^k}{\partial \bar{u}^n}\frac{\partial u^m}{\partial \bar{u}^j}\frac{\partial \bar{u}^n}{\partial u^k}\right)\frac{\partial \bar{u}^j}{\partial u^m} \\ &= \left[T^i_{,k} + T^m \Gamma^i_{km}\right] \equiv T^i\big|_k\end{aligned} \tag{2.202g}$$

Substituting Eqs. (2.202f) and (2.202g) into Eq. (2.202e), one obtains Eq. (2.202a).

$$\begin{aligned} T^i\big|_k &= \bar{T}^j\big|_n \frac{\partial u^i}{\partial \bar{u}^j}\frac{\partial \bar{u}^n}{\partial u^k} \\ \Rightarrow \bar{T}^j\big|_n &= T^i\big|_k \frac{\partial \bar{u}^j}{\partial u^i}\frac{\partial u^k}{\partial \bar{u}^n} \text{ (q.e.d.)}\end{aligned}$$

(b) **Covariant first-order tensors with components T_i**

The covariant first-order tensor (vector) **T** can be written in the contravariant basis.

$$\mathbf{T} = T_i \mathbf{g}^i \tag{2.203}$$

Using Eq. (2.189), the partial derivative of the tensor **T** results in

$$\begin{aligned}\mathbf{T}_{,j} &= \left(T_i \mathbf{g}^i\right)_{,j} = T_{i,j}\mathbf{g}^i + T_i \mathbf{g}^i_{,j} \\ &= T_{i,j}\mathbf{g}^i - T_i \Gamma^i_{jk}\mathbf{g}^k\end{aligned} \tag{2.204}$$

The partial derivative of the covariant tensor component T_i with respect to u^j in Eq. (2.204) can be defined as

$$T_{i,j} \equiv \frac{\partial T_i}{\partial u^j} \tag{2.205}$$

Interchanging i with k in the second term in the RHS of Eq. (2.204), one obtains

$$T_i \Gamma^i_{jk}\mathbf{g}^k \equiv T_k \Gamma^k_{ji}\mathbf{g}^i = T_k \Gamma^k_{ij}\mathbf{g}^i \tag{2.206}$$

Substituting Eq. (2.206) into Eq. (2.204), one obtains the derivative of the first-order tensor component (vector) **T** with respect to u^j.

$$\begin{aligned}\mathbf{T}_{,j} &= T_{i,j}\mathbf{g}^i - T_k \Gamma^k_{ij}\mathbf{g}^i \\ &= \left(T_{i,j} - \Gamma^k_{ij} T_k\right)\mathbf{g}^i \\ &\equiv T_i\big|_j \mathbf{g}^i\end{aligned} \tag{2.207}$$

Therefore, the covariant derivative of the tensor component T_i with respect to u^j can be defined as

$$T_{\mathrm{i}}\big|_j = T_{i,j} - \Gamma_{ij}^k T_k = \mathbf{T}_{,j} \cdot \mathbf{g}_i \tag{2.208}$$

The covariant derivative of the first-order tensor component T_i is transformed in the new barred coordinates $\{\bar{u}^{\mathrm{i}}\}$, similar to Eq. (2.202a), cf. (Nayak 2012; De et al. 2012).

$$\begin{aligned} T_k|_l &= \bar{T}_i\big|_j \frac{\partial \bar{u}^i}{\partial u^k} \frac{\partial \bar{u}^j}{\partial u^l} \\ \Rightarrow \bar{T}_i\big|_j &= T_k|_l \frac{\partial u^k}{\partial \bar{u}^i} \frac{\partial u^l}{\partial \bar{u}^j} \end{aligned} \tag{2.209}$$

(c) **Second-order tensors**

Second-order tensors can be written in different expressions with covariant and contravariant bases.

$$\mathbf{T} = T_{\mathrm{ij}} \mathbf{g}^{\mathrm{i}} \mathbf{g}^{\mathrm{j}} = T^{\mathrm{ij}} \mathbf{g}_{\mathrm{i}} \mathbf{g}_{\mathrm{j}} = T_{\mathrm{i}}^{\mathrm{j}} \mathbf{g}^{\mathrm{i}} \mathbf{g}_{\mathrm{j}} = T_{\mathrm{j}}^{\mathrm{i}} \mathbf{g}_{\mathrm{i}} \mathbf{g}^{\mathrm{j}} \tag{2.210}$$

Similarly, the covariant derivatives with respect to u^k of the second-order tensor components of **T** can be calculated as, cf. (Klingbeil 1966; Nayak 2012; De et al. 2012)

$$\begin{aligned} T_{ij}|_k &= T_{ij,k} - \Gamma_{ik}^m T_{mj} - \Gamma_{jk}^m T_{im} \\ T^{ij}|_k &= T^{\mathrm{ij}}_{,\mathrm{k}} + \Gamma_{km}^i T^{mj} + \Gamma_{km}^j T^{im} \\ T_j^i|_k &= T_{j,k}^i + \Gamma_{km}^i T_j^m - \Gamma_{jk}^m T_m^i \\ T_i^j|_k &= T_{i,k}^j - \Gamma_{ik}^m T_m^j + \Gamma_{km}^j T_i^m \end{aligned} \tag{2.211a}$$

where $T_{ij,k}, T^{ij}_{,k}$, and $T^i_{j,k}$ are the partial derivatives with respect to u^k of the covariant, contravariant, and mixed tensor components. Note that they are different to the covariant derivatives of the tensor components, as defined in Eq. (2.211a).

In the coordinate transformation from the curvilinear coordinates $\{u^i\}$ to the new barred curvilinear coordinates $\{\bar{u}^\alpha\}$, the covariant derivative of the covariant second-order tensor with respect to $\bar{u}^\gamma$ can be calculated using the chain rule of differentiation, similar to Eq. (2.202a), cf. (Nayak 2012; De et al. 2012).

$$\bar{T}_{\alpha\beta}\big|_\gamma = T_{ij}\big|_k \frac{\partial u^i}{\partial \bar{u}^\alpha} \frac{\partial u^j}{\partial \bar{u}^\beta} \frac{\partial u^k}{\partial \bar{u}^\gamma} \tag{2.211b}$$

where the partial derivatives $\bar{u}^{\alpha}_{,i}$ are called the shift tensor between two coordinate systems. This relation in Eq. (2.211b) is the chain rule of the covariant derivatives of the second-order tensors in the coordinate transformation.

Analogously, the covariant derivatives of the second-order tensors of different types in the new barred curvilinear coordinates are calculated using the shift tensors.

$$\begin{aligned}
\bar{T}_{\alpha\beta}\big|_{\gamma} &= T_{ij}\big|_{k}\frac{\partial u^i}{\partial \bar{u}^{\alpha}}\frac{\partial u^j}{\partial \bar{u}^{\beta}}\frac{\partial u^k}{\partial \bar{u}^{\gamma}}\,; \\
\bar{T}^{\alpha\beta}\big|_{\gamma} &= T^{ij}\big|_{k}\frac{\partial \bar{u}^{\alpha}}{\partial u^i}\frac{\partial \bar{u}^{\beta}}{\partial u^j}\frac{\partial u^k}{\partial \bar{u}^{\gamma}}\,; \\
\bar{T}^{\alpha}_{\beta}\big|_{\gamma} &= T^{i}_{j}\big|_{k}\frac{\partial \bar{u}^{\alpha}}{\partial u^i}\frac{\partial u^j}{\partial \bar{u}^{\beta}}\frac{\partial u^k}{\partial \bar{u}^{\gamma}}\,.
\end{aligned} \tag{2.211c}$$

2.5.10 Riemann–Christoffel Tensor

The Riemann–Christoffel tensor is closely related to the Gaussian curvature of the surface in differential geometry that will be discussed in Chap. 3.

At first, let us look into the second covariant derivative of an arbitrary first-order tensor of which the first covariant derivative with respect to u^j has been derived in Eq. (2.208)

$$T_i\big|_j = T_{i,j} - \Gamma^k_{ij}T_k \tag{2.212}$$

Obviously, the covariant derivative $T_i|_j$ is a second-order tensor component.

Differentiating $T_i|_j$ with respect to u^k, the first covariant derivative of the second-order tensor (component) $T_i|_j$ is the second covariant derivative of an arbitrary first-order tensor (component) T_i. This second covariant derivative has been given from Eq. (2.211a) (Klingbeil 1966).

$$\begin{aligned}
T_i\big|_{jk} &\equiv (T_i\big|_j)\big|_k \\
&= (T_i\big|_j)_{,k} - \Gamma^m_{ik}T_m\big|_j - \Gamma^m_{jk}T_i\big|_m \\
&= T_i\big|_{j,k} - \Gamma^m_{ik}T_m\big|_j - \Gamma^m_{jk}T_i\big|_m
\end{aligned} \tag{2.213}$$

Equation (2.212) delivers the relations of

$$T_i\big|_{j,k} = T_{i,jk} - \left(\Gamma^m_{ij,k}T_m + \Gamma^m_{ij}T_{m,k}\right) \tag{2.214a}$$

$$\Gamma^m_{ik}T_m\big|_j = \Gamma^m_{ik}\left(T_{m,j} - \Gamma^n_{mj}T_n\right) \tag{2.214b}$$

$$\Gamma^m_{jk} T_i|_m = \Gamma^m_{jk}\left(T_{i,m} - \Gamma^n_{im} T_n\right) \tag{2.214c}$$

Inserting Eqs. (2.214a), (2.214b), and (2.214c) into Eq. (2.213), one obtains the second covariant derivative of T_i.

$$\begin{aligned} T_i\big|_{jk} &= T_i\big|_{j,k} - \Gamma^m_{ik} T_m\big|_j - \Gamma^m_{jk} T_i|_m \\ &= T_{i,jk} - (\Gamma^m_{ij,k} T_m + \Gamma^m_{ij} T_{m,k}) \\ &\quad - \Gamma^m_{ik}(T_{m,j} - \Gamma^n_{mj} T_n) - \Gamma^m_{jk}(T_{i,m} - \Gamma^n_{im} T_n) \\ &= T_{i,jk} - \Gamma^m_{ij,k} T_m - \Gamma^m_{ij} T_{m,k} \\ &\quad - \Gamma^m_{ik} T_{m,j} + \Gamma^m_{ik}\Gamma^n_{mj} T_n - \Gamma^m_{jk} T_{i,m} + \Gamma^m_{jk}\Gamma^n_{im} T_n \end{aligned} \tag{2.215}$$

where the second partial derivative of T_i is symmetric with respect to j and k:

$$T_{i,jk} \equiv \frac{\partial^2 T_i}{\partial u^j \partial u^k} = \frac{\partial^2 T_i}{\partial u^k \partial u^j} \equiv T_{i,kj} \tag{2.216}$$

Interchanging the indices j with k in Eq. (2.215), one obtains

$$\begin{aligned} T_i\big|_{kj} &= T_{i,kj} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} T_{m,j} \\ &\quad - \Gamma^m_{ij} T_{m,k} + \Gamma^m_{ij}\Gamma^n_{mk} T_n - \Gamma^m_{kj} T_{i,m} + \Gamma^m_{kj}\Gamma^n_{im} T_n \end{aligned} \tag{2.217}$$

Using the symmetry properties given in Eqs. (2.164) and (2.216), Eq. (2.217) can be rewritten as

$$\begin{aligned} T_i\big|_{kj} &= T_{i,jk} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} T_{m,j} \\ &\quad - \Gamma^m_{ij} T_{m,k} + \Gamma^m_{ij}\Gamma^n_{mk} T_n - \Gamma^m_{jk} T_{i,m} + \Gamma^m_{jk}\Gamma^n_{im} T_n \end{aligned} \tag{2.218}$$

In a flat space, the second covariant derivatives in Eqs. (2.215) and (2.218) are identical. However, they are not equal in a curved space because of its surface curvature. The difference of both second covariant derivatives is proportional to the curvature tensor. Subtracting Eq. (2.215) from Eq. (2.218), the curvature tensor results in

$$\begin{aligned} T_i\big|_{jk} - T_i\big|_{kj} &= (\Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk}) T_n \\ &\equiv R^n_{ijk} T_n \end{aligned} \tag{2.219}$$

The Riemann–Christoffel tensor (Riemann curvature tensor) can be expressed as

$$R^n_{ijk} \equiv \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk} \tag{2.220}$$

Straightforwardly, the Riemann–Christoffel tensor is a fourth-order tensor with respect to the indices of i, j, k, and n. They contain 81 (=3^4) components in a three-dimensional space.

In Eq. (2.220), the partial derivatives of the Christoffel symbols are defined by

$$\Gamma^n_{ik,j} = \frac{\partial \Gamma^n_{ik}}{\partial u^j}; \quad \Gamma^n_{ij,k} = \frac{\partial \Gamma^n_{ij}}{\partial u^k} \tag{2.221}$$

Furthermore, the covariant Riemann curvature tensor of fourth order is defined by the Riemann–Christoffel tensor and covariant metric coefficients.

$$R_{lijk} = g_{ln} R^n_{ijk} \Leftrightarrow R^n_{ijk} = g^{ln} R_{lijk} \tag{2.222}$$

The Riemann curvature tensor has four following properties using the relation given in Eq. (2.222) (Nayak 2012):

- First skew symmetry with respect to l and i:

$$R_{lijk} = -R_{iljk} \tag{2.223}$$

- Second skew symmetry with respect to j and k:

$$\begin{aligned} R_{lijk} &= -R_{likj}; \\ R^n_{ijk} &= -R^n_{ikj} \end{aligned} \tag{2.224}$$

- Block symmetry with respect to two pairs (l, i) and (j, k):

$$R_{lijk} = R_{jkli} \tag{2.225}$$

- Cyclic property in i, j, k:

$$\begin{aligned} R_{lijk} + R_{ljki} + R_{lkij} &= 0; \\ R^n_{ijk} + R^n_{jki} + R^n_{kij} &= 0 \end{aligned} \tag{2.226}$$

Equation (2.226) is called the Bianchi first identity.

Resulting from these properties, there are six components of R_{lijk} in the three-dimensional space as follows (Klingbeil 1966):

$$R_{lijk} = R_{3131}, R_{3132}, R_{3232}, R_{1212}, R_{3112}, R_{3212} \tag{2.227}$$

In Cartesian coordinates, all Christoffel symbols equal zero according to Eq. (2.184). Therefore, the Riemann–Christoffel tensor, as given in Eq. (2.220) must be equal to zero.

$$\begin{aligned} R^n_{ijk} &= \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk} \\ &= 0 \end{aligned} \tag{2.228}$$

Thus, the Riemann surface curvature tensor can be written as

$$R_{lijk} = g_{\ln} R^n_{ijk} = 0 \tag{2.229}$$

In this case, Euclidean N-space with orthonormal Cartesian coordinates is considered as a flat space because the Riemann curvature tensor there equals zero.

In the following section, the Riemann curvature tensor can be calculated from the Christoffel symbols of first and second kinds.

From Eq. (2.222), the Riemann curvature tensor can be rewritten as

$$R_{hijk} = g_{hl} R^l_{ijk}$$

Using Eq. (2.220), the Riemann curvature tensor results in

$$\begin{aligned} R_{hijk} &= g_{hn} R^n_{ijk} \\ &= g_{hn}(\Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk}) \\ &= \frac{\partial(g_{hn}\Gamma^n_{ik})}{\partial u^j} - \Gamma^n_{ik} g_{hn,j} - \frac{\partial(g_{hn}\Gamma^n_{ij})}{\partial u^k} + \Gamma^n_{ij} g_{hn,k} + g_{hn}\Gamma^m_{ik}\Gamma^n_{mj} - g_{hn}\Gamma^m_{ij}\Gamma^n_{mk} \\ &= \frac{\partial(g_{hn}\Gamma^n_{ik})}{\partial u^j} - \Gamma^n_{ik} g_{hn,j} - \frac{\partial(g_{hn}\Gamma^n_{ij})}{\partial u^k} + \Gamma^n_{ij} g_{hn,k} + \Gamma^m_{ik}\Gamma_{mjh} - \Gamma^m_{ij}\Gamma_{mkh} \end{aligned} \tag{2.230a}$$

Changing the index m into n in both last terms on the RHS of Eq. (2.230a), one obtains

$$\begin{aligned} R_{hijk} &= \frac{\partial(g_{hn}\Gamma^n_{ik})}{\partial u^j} - \Gamma^n_{ik} g_{hn,j} - \frac{\partial(g_{hn}\Gamma^n_{ij})}{\partial u^k} + \Gamma^n_{ij} g_{hn,k} + \Gamma^n_{ik}\Gamma_{njh} - \Gamma^n_{ij}\Gamma_{nkh} \\ &= \frac{\partial(g_{hn}\Gamma^n_{ik})}{\partial u^j} - \frac{\partial(g_{hn}\Gamma^n_{ij})}{\partial u^k} - \Gamma^n_{ik}(g_{hn,j} - \Gamma_{njh}) + \Gamma^n_{ij}(g_{hn,k} - \Gamma_{nkh}) \end{aligned} \tag{2.230b}$$

Using Eq. (2.192) for the first-kind Christoffel symbols in Eq. (2.230b), the Riemann curvature tensor becomes

$$\begin{aligned} R_{hijk} = g_{hn} R^n_{ijk} &= \frac{\partial \Gamma_{ikh}}{\partial u^j} - \frac{\partial \Gamma_{ijh}}{\partial u^k} \\ &\quad - \Gamma^n_{ik}(\Gamma_{hjn} + \underline{\Gamma_{njh}} - \underline{\Gamma_{njh}}) + \Gamma^n_{ij}(\Gamma_{hkn} + \underline{\Gamma_{nkh}} - \underline{\Gamma_{nkh}}) \\ &= \Gamma_{ikh,j} - \Gamma_{ijh,k} - \Gamma^n_{ik}\Gamma_{hjn} + \Gamma^n_{ij}\Gamma_{hkn} \end{aligned} \tag{2.230c}$$

2.5.11 Ricci's Lemma

The covariant derivative of the metric covariant coefficient g_{ij} with respect to u^k results from Eq. (2.211a) changing T_{ij} into g_{ij}. Then, using Eq. (2.193), one obtains

$$\begin{aligned} g_{ij}|_k &= \frac{\partial g_{ij}}{\partial u^k} - \left(g_{mj}\Gamma^m_{ik} + g_{in}\Gamma^n_{jk} \right) \\ &= g_{ij,k} - g_{ij,k} = 0 \rightarrow \text{(q.e.d.)} \end{aligned} \tag{2.231a}$$

Therefore,

$$g_{ij,k} \equiv \frac{\partial g_{ij}}{\partial u^k} = g_{mj}\Gamma^m_{ik} + g_{in}\Gamma^n_{jk} \tag{2.231b}$$

The Kronecker delta is the product of the covariant and contravariant metric coefficients:

$$\delta^j_l = g_{li} g^{ij}$$

The partial derivative with respect to u^k of the Kronecker delta (invariant) equals zero and can be written as

$$\begin{aligned} \delta^j_{l,k} &= \left(g_{li} g^{ij} \right)_{,k} = \frac{\partial g_{li}}{\partial u^k} g^{ij} + g_{li} \frac{\partial g^{ij}}{\partial u^k} \\ &= g_{li,k} g^{ij} + g_{li} g^{ij}_{,k} \\ &= 0 \end{aligned} \tag{2.232a}$$

Multiplying Eq. (2.232a) by g^{lm}, one obtains

$$\begin{aligned} & g^{ij} g^{lm} g_{li,k} + \delta^m_i g^{ij}_{,k} = 0 \\ & \Rightarrow g^{mj}_{,k} = -g^{ij} g^{lm} g_{li,k} \end{aligned} \tag{2.232b}$$

Interchanging the indices, it results in

$$\begin{aligned} & g^{mj}_{,k} = -g^{ij} g^{lm} g_{li,k} \\ & \Rightarrow g^{ij}_{,k} = -g^{mi} g^{nj} g_{mn,k} \end{aligned} \tag{2.232c}$$

Using Eq. (2.211a), the covariant derivative of the metric contravariant coefficient g^{ij} with respect to u^k can be written as

$$
\begin{aligned}
g^{ij}\big|_k &= \frac{\partial g^{ij}}{\partial u^k} + g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km} \\
&= g^{ij}_{,k} + g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}
\end{aligned}
\tag{2.233a}
$$

Substituting Eqs. (2.231b), (2.232c), and (2.233a), one obtains after interchanging the indices

$$
\begin{aligned}
g^{ij}\big|_k &= g^{ij}_{,k} + \left(g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\right) \\
&= -g^{mi}g^{nj}g_{mn,k} + \left(g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\right) \\
&= -g^{mi}\Gamma^j_{mk} - g^{nj}\Gamma^i_{nk} + \left(g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\right) \\
&= -g^{im}\Gamma^j_{km} - g^{mj}\Gamma^i_{km} + \left(g^{mj}\Gamma^i_{km} + g^{im}\Gamma^j_{km}\right) \\
&= 0 \rightarrow \text{(q.e.d.)}
\end{aligned}
\tag{2.233b}
$$

Note that Eqs. (2.231a) and (2.233b) are known as *Ricci's lemma*.

2.5.12 Derivative of the Jacobian

In the following section, the derivative of the Jacobian J can be calculated and its result is very useful in the Nabla operator [cf. (Nayak 2012; De et al. 2012)].

The determinant of the metric coefficient tensor is given from Eq. (2.17):

$$
\det(g_{ij}) = \begin{vmatrix} g_{11} & g_{12} & \cdot & g_{1N} \\ g_{21} & g_{22} & \cdot & g_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ g_{N1} & g_{N2} & \cdot & g_{NN} \end{vmatrix} = g = J^2 > 0
\tag{2.234}
$$

The contravariant metric coefficient g^{ij} results from the cofactor G^{ij} of the covariant metric coefficient g_{ij} and the determinant g.

$$
g^{ij} = \frac{G^{ij}}{g} \Rightarrow G^{ij} = g g^{ij}
\tag{2.235}
$$

Differentiating both sides of Eq. (2.234) with respect to u^k, one obtains

$$
\frac{\partial g}{\partial u^k} = \frac{\partial g_{ij}}{\partial u^k} G^{ij} \quad \text{for } i,j = 1,2,\ldots,N
\tag{2.236}
$$

Prove Eq. (2.236):

$$\frac{\partial g}{\partial u^k} = \begin{vmatrix} \frac{\partial g_{11}}{\partial u^k} & \frac{\partial g_{12}}{\partial u^k} & \cdot & \frac{\partial g_{1N}}{\partial u^k} \\ g_{21} & g_{22} & \cdot & g_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ g_{N1} & g_{N2} & \cdot & g_{NN} \end{vmatrix} + \ldots + \begin{vmatrix} g_{11} & g_{12} & \cdot & g_{1N} \\ g_{21} & g_{22} & \cdot & g_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial g_{N1}}{\partial u^k} & \frac{\partial g_{N2}}{\partial u^k} & \cdot & \frac{\partial g_{NN}}{\partial u^k} \end{vmatrix}$$
$$= \frac{\partial g_{11}}{\partial u^k} G^{11} + \frac{\partial g_{12}}{\partial u^k} G^{12} + \ldots + \ldots + \frac{\partial g_{NN}}{\partial u^k} G^{NN}$$
$$= \frac{\partial g_{ij}}{\partial u^k} G^{ij} \, for \; i,j = 1,2,\ldots,N \rightarrow \text{(q.e.d.)} \tag{2.237}$$

Substituting Eq. (2.235) into Eq. (2.236), it gives

$$\begin{aligned} \frac{\partial g}{\partial u^k} &= \frac{\partial g_{ij}}{\partial u^k} G^{ij} \\ &= \frac{\partial g_{ij}}{\partial u^k} g g^{ij} \\ &\equiv g_{ij,k} g g^{ij} \end{aligned} \tag{2.238}$$

Inserting Eq. (2.231b) into Eq. (2.238), one obtains

$$\begin{aligned} \frac{\partial g}{\partial u^k} &= g_{ij,k} g g^{ij} \\ &= \left(g_{mj} \Gamma^m_{ik} + g_{in} \Gamma^n_{jk} \right) g g^{ij} \\ &= g \left(\delta^i_m \Gamma^m_{ik} + \delta^j_n \Gamma^n_{jk} \right) \\ &= g \left(\Gamma^i_{ik} + \Gamma^j_{jk} \right) = 2g \Gamma^i_{ik} \end{aligned} \tag{2.239}$$

Using the chain rule of differentiation, the Christoffel symbol in Eq. (2.239) can be expressed in the Jacobian J.

$$\begin{aligned} \Gamma^i_{ik} &= \frac{1}{2g} \frac{\partial g}{\partial u^k} = \frac{\partial (\ln \sqrt{g})}{\partial u^k} \\ &= \frac{\partial (\ln J)}{\partial u^k} = \frac{1}{J} \frac{\partial J}{\partial u^k} \end{aligned} \tag{2.240}$$

Prove that $R^i_{ijk} = 0$

Using Eq. (2.220) for $n = i$, the Riemann–Christoffel tensor can be written as

$$R^i_{ijk} = \Gamma^i_{ik,j} - \Gamma^i_{ij,k} + \Gamma^m_{ik} \Gamma^i_{mj} - \Gamma^m_{ij} \Gamma^i_{mk}$$

Interchanging j with k in the last term on the RHS of the equation, one obtains

$$\begin{aligned} R^i_{ijk} &= \Gamma^i_{ik,j} - \Gamma^i_{ij,k} + \Gamma^m_{ik}\Gamma^i_{mj} - \Gamma^m_{ik}\Gamma^i_{mj} \\ &= \Gamma^i_{ik,j} - \Gamma^i_{ij,k} \end{aligned}$$

Using Eq. (2.240), the above Riemann–Christoffel tensor can be rewritten as

$$\begin{aligned} R^i_{ijk} &= \Gamma^i_{ik,j} - \Gamma^i_{ij,k} \\ &= \frac{\partial(\ln J)}{\partial u^j \partial u^k} - \frac{\partial(\ln J)}{\partial u^k \partial u^j} \\ &= 0 \rightarrow \ (\text{q.e.d.}) \end{aligned} \tag{2.241}$$

2.5.13 Ricci Tensor

Both Ricci and Einstein tensors are very useful mathematical tools in the relativity theory. Note that tensors using in the relativity fields have been mostly written in the abstract index notation defined by Penrose (Penrose 2005). This index notation uses the indices to express the tensor types, rather than their covariant components in the basis $\{\mathbf{g}^i\}$. The first-kind Ricci tensor results from the index contraction of k and n for $n = k$ of the Riemann–Christoffel tensor, as given in Eq. (2.220).

$$\begin{aligned} R_{ij} &\equiv R^k_{ijk} \\ &= \frac{\partial \Gamma^k_{ik}}{\partial u^j} - \frac{\partial \Gamma^k_{ij}}{\partial u^k} - \Gamma^r_{ij}\Gamma^k_{rk} + \Gamma^r_{ik}\Gamma^k_{rj} \end{aligned} \tag{2.242}$$

The second-kind Ricci tensor can be defined as

$$\begin{aligned} R^i_j &\equiv g^{ik} R_{kj} \\ &= g^{ik}\left(\frac{\partial \Gamma^m_{km}}{\partial u^j} - \frac{\partial \Gamma^m_{kj}}{\partial u^m} - \Gamma^m_{kj}\Gamma^n_{mn} + \Gamma^m_{kn}\Gamma^n_{mj} \right) \end{aligned} \tag{2.243}$$

Using the Christoffel symbol in Eq. (2.240), we obtain

$$\Gamma^j_{ij} = \frac{1}{J}\frac{\partial J}{\partial u^i} = \frac{\partial(\ln J)}{\partial u^i}, \tag{2.244}$$

The first-kind Ricci tensor can be rewritten as

$$\begin{aligned} R_{ij} &= \frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \frac{\partial \Gamma_{ij}^k}{\partial u^k} - \Gamma_{ij}^k \frac{\partial(\ln J)}{\partial u^k} + \Gamma_{ik}^r \Gamma_{rj}^k \\ &= \frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \left(\frac{\partial \Gamma_{ij}^k}{\partial u^k} + \Gamma_{ij}^k \frac{\partial(\ln J)}{\partial u^k} \right) + \Gamma_{ik}^r \Gamma_{rj}^k \\ &= \frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \frac{1}{J} \left(J \frac{\partial \Gamma_{ij}^k}{\partial u^k} + \Gamma_{ij}^k \frac{\partial J}{\partial u^k} \right) + \Gamma_{ik}^r \Gamma_{rj}^k \\ &= \frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \frac{1}{J} \frac{\partial (J \Gamma_{ij}^k)}{\partial u^k} + \Gamma_{ik}^r \Gamma_{rj}^k \end{aligned} \tag{2.245}$$

Interchanging i with j in Eq. (2.245), one obtains the relation of

$$\begin{aligned} R_{ji} &= \frac{\partial^2(\ln J)}{\partial u^j \partial u^i} - \frac{1}{J} \frac{\partial (J \Gamma_{ji}^k)}{\partial u^k} + \Gamma_{jk}^r \Gamma_{ri}^k \\ &= \frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \frac{1}{J} \frac{\partial (J \Gamma_{ij}^k)}{\partial u^k} + \Gamma_{ir}^k \Gamma_{kj}^r \\ &= R_{ij} \end{aligned} \tag{2.246}$$

This result indicates that the first-kind Ricci tensor is symmetric with respect to i and j.

Substituting Eq. (2.244) into Eq. (2.243), the second-kind Ricci tensor results in

$$R_j^i = g^{ik} \left(\frac{\partial^2(\ln J)}{\partial u^k \partial u^j} - \frac{1}{J} \frac{\partial (J \Gamma_{kj}^m)}{\partial u^m} + \Gamma_{kn}^m \Gamma_{mj}^n \right) \tag{2.247}$$

The Ricci curvature R can be defined as

$$\begin{aligned} R &\equiv R_i^i \\ &= g^{ij} R_{ji} = g^{ij} R_{ij} \end{aligned} \tag{2.248}$$

Substituting Eq. (2.246) into Eq. (2.248), the Ricci curvature results in

$$R = g^{ij} \left(\frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \frac{1}{J} \frac{\partial (J \Gamma_{ij}^k)}{\partial u^k} + \Gamma_{ir}^k \Gamma_{kj}^r \right) \tag{2.249}$$

2.5.14 Einstein Tensor

The Einstein tensor is defined by the second-kind Ricci tensor, Kronecker delta, and the Ricci curvature.

$$G_j^i \equiv R_j^i - \frac{1}{2}\delta_j^i R \tag{2.250a}$$

The Einstein tensor is a mixed second-order tensor and can be written as

$$G_j^i = g^{ik} G_{kj} \tag{2.250b}$$

Using the tensor contraction rules, the covariant Einstein tensor results in

$$\begin{aligned} G_{ij} &= g_{ik} G_j^k = g_{ik}\left(R_j^k - \frac{1}{2}\delta_j^k R\right) \\ &= R_{ij} - \frac{1}{2} g_{ij} R \\ &= R_{ji} - \frac{1}{2} g_{ji} R \\ &= G_{ji} \end{aligned} \tag{2.251}$$

This result proves that the covariant Einstein tensor is symmetric due to the symmetry of the Ricci tensor.

The Bianchi first identity in Eq. (2.226) gives

$$\begin{aligned} R_{lijk} + R_{ljki} + R_{lkij} = 0; \\ R^n_{ijk} + R^n_{jki} + R^n_{kij} = 0 \end{aligned} \tag{2.252}$$

Differentiating covariantly Eq. (2.252) with respect to u^m, u^k, and u^l and then multiplying it by the covariant metric coefficients g_{in}, one obtains the *Bianchi second identity*, cf. Nayak (2012), De et al. (2012), Lee (2000), and Helgason (1978).

$$\begin{aligned} R^n_{jkl}\Big|_m + R^n_{jlm}\Big|_k + R^n_{jmk}\Big|_l = 0.g_{in} \\ \Rightarrow R_{ijkl}\big|_m + R_{ijlm}\big|_k + R_{ijmk}\big|_l = 0 \end{aligned} \tag{2.253}$$

Due to skew symmetry of the covariant Riemann curvature tensors, as discussed in Eqs. (2.223) and (2.224), Eq. (2.253) can be rewritten as

$$R_{ijkl}\big|_m - R_{ijml}\big|_k - R_{jimk}\big|_l = 0 \tag{2.254}$$

Multiplying Eq. (2.254) by $g^{il}g^{jk}$ and using the tensor contraction rules (cf. Sect. 2.3.5), one obtains

$$\begin{aligned}
&R_{ijkl}\big|_m - R_{ijml}\big|_k - R_{jimk}\big|_l = 0 \;\Leftrightarrow\\
&g^{il}g^{jk}R_{ijkl}\big|_m - g^{il}g^{jk}R_{ijml}\big|_k - g^{il}g^{jk}R_{jimk}\big|_l\\
&= g^{jk}R_{jk}\big|_m - g^{jk}R_{jm}\big|_k - g^{il}R_{im}\big|_l\\
&= R\big|_m - R^k_m\big|_k - R^l_m\big|_l \quad (l \to k)\\
&= R\big|_m - 2R^k_m\big|_k\\
&= 0
\end{aligned}$$

Thus,

$$R^k_m\big|_k = \frac{1}{2}R\big|_m \tag{2.255}$$

Using Eq. (2.232a) and the symmetry of the Christoffel symbols, the covariant derivative of the Kronecker symbol with respect to u^k is equal to zero.

$$\begin{aligned}
\delta^i_j\Big|_k &= \delta^i_{j,k} + \Gamma^i_{km}\delta^m_j - \Gamma^m_{jk}\delta^i_m\\
&= \delta^i_{j,k} + \left(\Gamma^i_{kj} - \Gamma^i_{jk}\right)\\
&= 0
\end{aligned} \tag{2.256}$$

Differentiating covariantly the Einstein tensor in Eq. (2.250a) with respect to u^k and using Eq. (2.256), one obtains the covariant derivative

$$\begin{aligned}
G^i_j\Big|_k &= \left(R^i_j - \frac{1}{2}\delta^i_j R\right)\Big|_k\\
&= R^i_j\Big|_k - \frac{1}{2}\left(\delta^i_j\Big|_k R + \delta^i_j R\Big|_k\right)\\
&= R^i_j\Big|_k - \frac{1}{2}\delta^i_j R\Big|_k
\end{aligned} \tag{2.257}$$

Changing the index i into k in Eq. (2.257) and using Eq. (2.255), the divergence of the Einstein tensor equals zero.

$$\begin{aligned}
&G^k_j\Big|_k = R^k_j\Big|_k - \frac{1}{2}\delta^k_j R\Big|_k = R^k_j\Big|_k - \frac{1}{2}R\big|_j = 0\\
&\Rightarrow \mathrm{Div}(\mathbf{G}) \equiv \nabla.\mathbf{G} = G^{\mathrm{k}}_{\mathrm{j}}\Big|_{\mathrm{k}}\mathbf{g}^{\mathrm{j}} = \mathbf{0} \quad \text{(q.e.d.)}
\end{aligned} \tag{2.258}$$

This result is very important and has been often used in the general relativity theories and other relativity fields.

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Chapter 3
Elementary Differential Geometry

3.1 Introduction

We consider an N-dimensional Riemannian manifold **M**, and let $\mathbf{g}_i$ be a basis at the point $P_i(u^1,\ldots,u^N)$ and $\mathbf{g}_j$ be another basis at the other point $P_j(u^1,\ldots,u^N)$. Note that each such basis may only exist in a local neighborhood of the respective points and not necessarily for the whole space. For each such point, we may construct an embedded affine tangential manifold. The N-tuple of coordinates are invariant in any chosen basis; however, its components on the coordinates change as the coordinate system varies. Therefore, the relating components have to be taken into account by the coordinate transformations.

3.2 Arc Length and Surface in Curvilinear Coordinates

Consider two points $P(u^1,\ldots,u^N)$ and $Q(u^1,\ldots,u^N)$ of an N-tuple of the coordinates $(u^1,\ldots,u^N)$ in the parameterized curve $\mathrm{C} \in \mathbf{R}^N$. The coordinates $(u^1,\ldots,u^N)$ can be assumed to be a function of the parameter λ that varies from $P(\lambda_1)$ to $Q(\lambda_2)$, as shown in Fig. 3.1.

The ***arc length*** $\mathbf{d}s$ between the points P and Q results from

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda}\right)^2 = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\lambda}\cdot\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\lambda} \tag{3.1}$$

where the derivative of the vector $\mathbf{r}(u, v)$ can be calculated by

$$\begin{aligned}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\lambda} &= \frac{\mathrm{d}(\mathbf{g}_i u^{\mathrm{i}})}{\mathrm{d}\lambda} = \mathbf{g}_i\left(\frac{\partial u^i}{\partial\lambda}\right) \\ &\equiv \mathbf{g}_i \dot{u}^i(\lambda); \quad \forall i = 1, 2\end{aligned} \tag{3.2}$$

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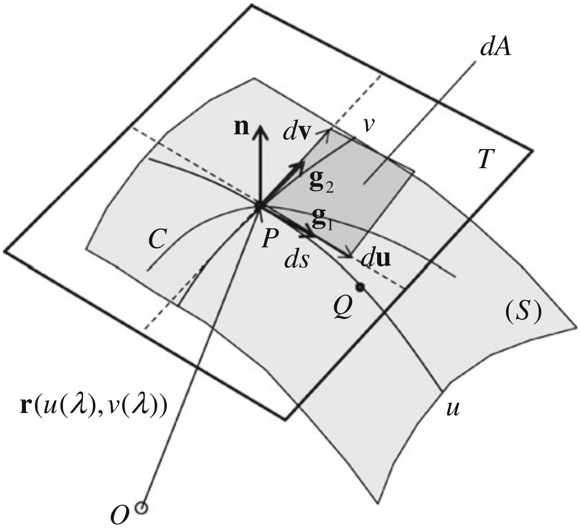

Fig. 3.1 Arc length and surface on the surface (S)

Substituting Eq. (3.2) into Eq. (3.1), one obtains the arc length PQ.

$$ds = \sqrt{\varepsilon(\mathbf{g}_i\dot{u}) \cdot (\mathbf{g}_j\dot{v})}\,d\lambda = \sqrt{\varepsilon g_{ij}\dot{u} \cdot \dot{v}}\,d\lambda \tag{3.3}$$

where ε (= ±1) is the functional indicator, which ensures that the square root always exists.

Therefore, the arc length of PQ is given by integrating Eq. (3.3) from the parameter λ_1 to the parameter λ_2.

$$s = \int_{\lambda_1}^{\lambda_2} \sqrt{\varepsilon g_{ij}\dot{u}(\lambda) \cdot \dot{v}(\lambda)}\,d\lambda \tag{3.4}$$

where the covariant metric coefficients g_{12} are defined by

$$\begin{aligned} g_{ij} &= g_{ji} = \mathbf{g}_i \cdot \mathbf{g}_j \\ &= \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v}; \quad i \equiv u, j \equiv v \end{aligned} \tag{3.5}$$

Thus, the metric coefficient tensor of the parameterized surface S is given in

$$\begin{aligned} \mathbf{M} = (g_{ij}) &= \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{r}_u\mathbf{r}_u & \mathbf{r}_u\mathbf{r}_v \\ \mathbf{r}_v\mathbf{r}_u & \mathbf{r}_v\mathbf{r}_v \end{bmatrix} \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} \end{aligned} \tag{3.6}$$

The area differential of the tangent plane T at the point P can be calculated by

$$\begin{aligned} \mathrm{d}A &= |\mathbf{du} \times \mathbf{dv}| \\ &= |\mathbf{g}_1 \mathrm{d}u \times \mathbf{g}_2 \mathrm{d}v| = |\mathbf{g}_1 \times \mathbf{g}_2| \mathrm{d}u\mathrm{d}v \end{aligned} \tag{3.7}$$

Using the Lagrange's identity, Eq. (3.7) becomes

$$\begin{aligned} \mathrm{d}A &= |\mathbf{g}_1 \times \mathbf{g}_2| \mathrm{d}u\mathrm{d}v \\ &= \sqrt{g_{11}g_{22} - (g_{12})^2}\, \mathrm{d}u\mathrm{d}v = \sqrt{\det(g_{ij})}\, \mathrm{d}u\mathrm{d}v \\ &= \sqrt{EG - F^2}\, \mathrm{d}u\mathrm{d}v \end{aligned} \tag{3.8}$$

Integrating Eq. (3.8), the area of the surface S results in

$$A = \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{\lambda_2} \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v \tag{3.9}$$

In the following section, the circumference at the equator and surface area of a sphere with a radius R are calculated (see Fig. 3.2).

The location vector of a given point $P(u(\lambda), v(\lambda))$ in the parameterized surface of the sphere (S) can be written as

$$\begin{aligned} &(S) : x^2 + y^2 + z^2 = R^2 \Rightarrow \\ &\mathbf{r}(\phi, \theta) = \begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix}; u \equiv \phi \in [0, \pi[;\ v \equiv \theta \in [0, 2\pi[\end{aligned} \tag{3.10}$$

The covariant bases can be calculated in

$$\begin{aligned} \mathbf{g}_u &= \frac{\partial \mathbf{r}}{\partial \phi} = \begin{pmatrix} R\cos\phi\cos\theta \\ R\cos\phi\sin\theta \\ -R\sin\phi \end{pmatrix}; \\ \mathbf{g}_v &= \frac{\partial \mathbf{r}}{\partial \theta} = \begin{pmatrix} -R\sin\phi\sin\theta \\ R\sin\phi\cos\theta \\ 0 \end{pmatrix} \end{aligned} \tag{3.11}$$

Thus, the metric coefficient tensor results from Eq. (3.11).

$$\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\phi \end{bmatrix} \tag{3.12}$$

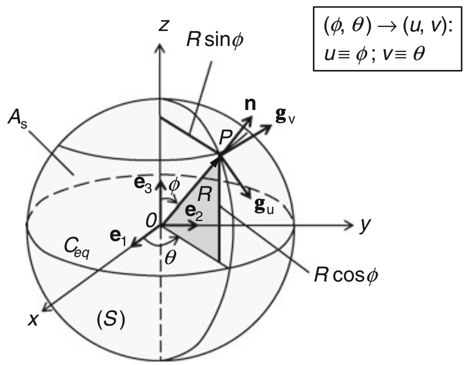

Fig. 3.2 Arc length and surface of a sphere (S)

The circumference at the equator is given at $u \equiv \phi = \pi/2$ and $v \equiv \theta(\lambda) = \lambda$.

$$C_{eq} = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{ij}\dot{u}(\lambda)\cdot\dot{v}(\lambda)}\mathrm{d}\lambda = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{11}\dot{u}^2 + 2g_{12}\dot{u}\dot{v} + g_{22}\dot{v}^2}\,\mathrm{d}\lambda \tag{3.13}$$

where $\dot{u} = 0\,;\dot{v} = 1$.

Therefore,

$$C_q = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{22}}\,\mathrm{d}\lambda = \int_0^{2\pi} R\sin\frac{\pi}{2}\mathrm{d}\lambda = 2\pi R \tag{3.14}$$

The surface area of the sphere can be computed according to Eq. (3.9).

$$\begin{aligned} A_S &= \int_{\lambda_1}^{\lambda_2}\int_{\lambda_1}^{\lambda_2} \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v = \int_0^{2\pi}\int_0^{\pi} \sqrt{EG - F^2}\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^{\pi} R^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta = -\int_0^{2\pi} R^2\cos\phi\big|_0^{\pi}\mathrm{d}\theta = 4\pi R^2 \end{aligned} \tag{3.15}$$

3.3 Unit Tangent and Normal Vector to Surface

The unit tangent vectors to the parameterized curves u and v at the point P have the same direction as the covariant bases $\mathbf{g}_1$ and $\mathbf{g}_2$. Both tangent vectors generate the tangent plane T to the differentiable Riemannian surface (S) at the point P. The unit normal vector $\mathbf{n}$ is perpendicular to the tangent plane T at the point P, as shown in Fig. 3.1.

The unit tangent vector **t** is defined, as given in Eq. (B.1a).

$$\mathbf{t}_i = \mathbf{g}_i^* \equiv \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}} = \frac{1}{\sqrt{g_{(ii)}}}\left(\frac{\partial \mathbf{r}}{\partial u^i}\right); \quad \forall i = 1,2$$
$$\Rightarrow \begin{cases} \mathbf{t}_1 = \mathbf{g}_1^* = \dfrac{1}{\sqrt{g_{11}}}\left(\dfrac{\partial \mathbf{r}}{\partial u}\right); & (1 \equiv u) \\ \mathbf{t}_2 = \mathbf{g}_2^* = \dfrac{1}{\sqrt{g_{22}}}\left(\dfrac{\partial \mathbf{r}}{\partial v}\right); & (2 \equiv v) \end{cases} \tag{3.16}$$

The unit normal vector is perpendicular to the unit tangent vectors at the point P and can be written as

$$\mathbf{n} = (\mathbf{t}_1 \times \mathbf{t}_2) = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1 \times \mathbf{g}_2|} = \frac{\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}}{\left|\dfrac{\partial \mathbf{r}}{\partial u} \times \dfrac{\partial \mathbf{r}}{\partial v}\right|} \tag{3.17}$$

Using Eq. (3.8), the unit normal vector can be rewritten as

$$\mathbf{n} = (\mathbf{t}_1 \times \mathbf{t}_2) = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1 \times \mathbf{g}_2|} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{\det(g_{ij})}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{EG - F^2}} \tag{3.18}$$

in which the cross product of $\mathbf{g}_1$ and $\mathbf{g}_2$ can be calculated by

$$\mathbf{g}_1 \times \mathbf{g}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \left(\dfrac{\partial \mathbf{r}}{\partial u}\right)_1 & \left(\dfrac{\partial \mathbf{r}}{\partial u}\right)_2 & \left(\dfrac{\partial \mathbf{r}}{\partial u}\right)_3 \\ \left(\dfrac{\partial \mathbf{r}}{\partial v}\right)_1 & \left(\dfrac{\partial \mathbf{r}}{\partial v}\right)_2 & \left(\dfrac{\partial \mathbf{r}}{\partial v}\right)_3 \end{vmatrix} = \varepsilon_{ijk}\left(\frac{\partial \mathbf{r}}{\partial u}\right)_i \left(\frac{\partial \mathbf{r}}{\partial v}\right)_j \mathbf{e}_k \tag{3.19}$$

where ε_{ijk} is the permutation symbol given in Eq. (A.5) in Appendix A.

The unit normal vector to the differentiable spherical surface (S) at the point P in Fig. 3.2 can be computed from Eq. (3.11).

$$\mathbf{g}_1 = \left(\frac{\partial \mathbf{r}}{\partial \phi}\right) = \begin{pmatrix} R\cos\phi\cos\theta \\ R\cos\phi\sin\theta \\ -R\sin\phi \end{pmatrix}; \quad \mathbf{g}_2 = \left(\frac{\partial \mathbf{r}}{\partial \theta}\right) = \begin{pmatrix} -R\sin\phi\sin\theta \\ R\sin\phi\cos\theta \\ 0 \end{pmatrix} \tag{3.20}$$

Therefore,

$$\mathbf{g}_1 \times \mathbf{g}_2 = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \end{vmatrix} = \begin{pmatrix} R^2\sin^2\phi\cos\theta \\ R^2\sin^2\phi\sin\theta \\ R^2\sin\phi\cos\phi \end{pmatrix} \tag{3.21}$$

Thus, the unit normal vector results from Eqs. (3.12), (3.18), and (3.21).

$$\mathbf{n} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\sqrt{EG - F^2}} = \frac{1}{R^2\sin\phi}\begin{pmatrix} R^2\sin^2\phi\cos\theta \\ R^2\sin^2\phi\sin\theta \\ R^2\sin\phi\cos\varphi \end{pmatrix} = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \tag{3.22}$$

Straightforwardly, the unit normal vector depends on each point $P(\phi, \theta)$ on the spherical surface (S) and has a vector length of 1.

3.4 The First Fundamental Form

The first and second fundamental forms of surfaces are two important characteristics in differential geometry as they are used to measure arc lengths and areas of surfaces, to identify isometric surfaces, and to find the extrema of surfaces. The Gaussian and mean curvatures of surfaces are based on both fundamental forms. Initially, the first fundamental form is examined in the following section.

Figure 3.3 displays the unit tangent vector **t** to the parameterized curve C at the point P in the differentiable surface (S). The unit normal vector **n** to the surface (S) at the point P is perpendicular to **t** and (S) at the point P.

Both unit tangent and normal vectors generate a Frenet orthonormal frame $\{\mathbf{t}, \mathbf{n}, (\mathbf{n} \times \mathbf{t})\}$ in which three unit vectors are orthogonal to each other, as shown in Fig. 3.3. The curvature vector **k** to the curve C can be rewritten as a linear combination of the normal curvature vector $\mathbf{k}_n$ and the geodesic curvature vector $\mathbf{k}_g$ at the point P in the Frenet frame.

$$\begin{aligned} \mathbf{k} = \mathbf{k}_n + \mathbf{k}_g &\Leftrightarrow \kappa\mathbf{n}_C = \kappa_n\mathbf{n} + \kappa_g(\mathbf{n} \times \mathbf{t}) \\ &\Rightarrow \kappa = \sqrt{\kappa_n^2(\lambda) + \kappa_g^2(\lambda)} \end{aligned} \tag{3.23}$$

where

κ is the curvature of the curve C at P

κ_n is the normal curvature of the surface (S) at P in the direction **t**

κ_g is the geodesic curvature of the surface (S) at P.

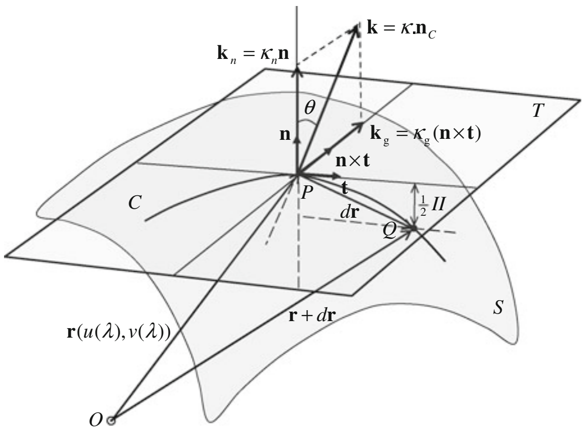

Fig. 3.3 Normal and geodesic curvatures of the surface S

The first fundamental form I of the surface (S) is defined by the arc length on the curve C in the surface (S).

$$\begin{aligned} I \equiv \mathrm{d}s^2 &= \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} \\ &= \left(\frac{\partial \mathbf{r}}{\partial u^i} \mathrm{d}u^i \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u^j} \mathrm{d}u^j \right) \end{aligned} \tag{3.24}$$

The first term in the RHS of Eq. (3.24) can be rewritten in the parameterized coordinate $u^i(\lambda)$.

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u^i} \mathrm{d}u^i &= \mathbf{g}_i \mathrm{d}u^i \\ &= \mathbf{g}_i \frac{\mathrm{d}u^i(\lambda)}{\mathrm{d}\lambda} \mathrm{d}\lambda = \mathbf{g}_i \dot{u}^i(\lambda) \mathrm{d}\lambda \end{aligned} \tag{3.25}$$

in which $\mathbf{g}_i$ is the covariant basis of the curvilinear coordinate u^i, as shown in Fig. 3.1.

Inserting Eq. (3.25) into Eq. (3.24), the first fundamental form results in

$$\begin{aligned} I &= \mathbf{g}_i \mathbf{g}_j \dot{u}^i(\lambda) \dot{u}^j(\lambda) \cdot \mathrm{d}\lambda^2 \\ &= g_{ij} \mathrm{d}u^i \mathrm{d}u^j \\ &= g_{11} \mathrm{d}u^2 + 2 g_{12} \mathrm{d}u \mathrm{d}v + g_{22} \mathrm{d}v^2 \end{aligned} \tag{3.26}$$

Using Eq. (3.6), the first fundamental form of the surface (S) can be rewritten as

$$I = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2 \tag{3.27a}$$

Therefore, the arc length ds can be rewritten as

$$\mathrm{d}s = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\mathrm{d}\lambda \tag{3.27b}$$

where E, F, and G are the covariant metric coefficients of the metric tensor **M**, as given in

$$\mathbf{M} = (g_{ij}) \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u\mathbf{r}_u & \mathbf{r}_u\mathbf{r}_v \\ \mathbf{r}_v\mathbf{r}_u & \mathbf{r}_v\mathbf{r}_v \end{bmatrix} \tag{3.28}$$

3.5 The Second Fundamental Form

The second fundamental form II is defined as twice of the projection of the arc length vector d**r** on the unit normal vector **n** of the parameterized surface (S) at the point P, as demonstrated in Fig. 3.3.

$$\frac{1}{2}\mathrm{II} \equiv \mathrm{d}\mathbf{r} \cdot \mathbf{n} \tag{3.29}$$

Using the Taylor's series for a vectorial function with two variables u and v, the differential of the arc length vector d**r**(u, v) can be written in the second order.

$$\begin{aligned} \mathrm{d}\mathbf{r} &= \left(\frac{\partial \mathbf{r}}{\partial u}\right)\mathrm{d}u + \left(\frac{\partial \mathbf{r}}{\partial v}\right)\mathrm{d}v \\ &\quad + \frac{1}{2}\left(\frac{\partial^2 \mathbf{r}}{\partial u^2}\mathrm{d}u^2 + 2\frac{\partial^2 \mathbf{r}}{\partial u \partial v}\mathrm{d}u\mathrm{d}v + \frac{\partial^2 \mathbf{r}}{\partial v^2}\mathrm{d}v^2\right) + O(d\mathbf{r}^3) \\ &\equiv \mathbf{r}_u\mathrm{d}u + \mathbf{r}_v\mathrm{d}v + \frac{1}{2}(\mathbf{r}_{uu}\mathrm{d}u^2 + 2\mathbf{r}_{uv}\mathrm{d}u\mathrm{d}v + \mathbf{r}_{vv}\mathrm{d}v^2) + O(d\mathbf{r}^3) \end{aligned} \tag{3.30}$$

Therefore, the second fundamental form can be computed as

$$II \approx 2(\mathbf{r}_u\mathbf{n}\mathrm{d}u + \mathbf{r}_v\mathbf{n}\mathrm{d}v) + (\mathbf{r}_{uu}\mathbf{n}\mathrm{d}u^2 + 2\mathbf{r}_{uv}\mathbf{n}\mathrm{d}u\mathrm{d}v + \mathbf{r}_{vv}\mathbf{n}\mathrm{d}v^2) \tag{3.31}$$

Due to the orthogonality of ($\mathbf{r}_u$, $\mathbf{r}_v$) and **n**, one obtains the inner products

$$\mathbf{r}_u\mathbf{n} = \mathbf{r}_v\mathbf{n} = 0 \tag{3.32}$$

where the covariant bases of the curvilinear coordinate (u, v) are shown in Fig. 3.1.

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \mathbf{g}_1; \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \mathbf{g}_2 \tag{3.33}$$

Substituting Eqs. (3.31) and (3.32), the second fundamental form results in

$$\begin{aligned} II &= \mathbf{r}_{uu}\mathbf{n}\mathrm{d}u^2 + 2\mathbf{r}_{uv}\mathbf{n}\mathrm{d}u\mathrm{d}v + \mathbf{r}_{vv}\mathbf{n}\mathrm{d}v^2 \\ &\equiv L\,\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2 \end{aligned} \tag{3.34}$$

in which L, M, and N are the elements of the Hessian tensor (Bär 2001; Chase 2012)

$$\mathbf{H} = (h_{\mathrm{ij}}) \equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu}\mathbf{n} & \mathbf{r}_{uv}\mathbf{n} \\ \mathbf{r}_{uv}\mathbf{n} & \mathbf{r}_{vv}\mathbf{n} \end{bmatrix} \tag{3.35a}$$

with

$$\mathbf{n} = (\mathbf{t}_1 \times \mathbf{t}_2) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{\det(g_{ij})}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{EG - F^2}} \tag{3.35b}$$

In the case of the projection equals zero, the second fundamental form II is also equal to zero. It gives the quadratic equation of du according to Eq. (3.34).

$$L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2 = 0 \tag{3.36}$$

Resolving Eq. (3.36) for du, one obtains the solution

$$\mathrm{d}u = \left(\frac{-M \pm \sqrt{(M^2 - LN)}}{L} \right) \mathrm{d}v \tag{3.37}$$

There are three cases for Eq. (3.37) with $L \neq 0$ (Klingbeil 1966; Kühnel 2013):

$(M^2 - LN) > 0$: two different solutions of du.
The surface (S) cuts the tangent plane T with two lines that intersect each other at the point P (hyperbolic point);
$(M^2 - LN) = 0$: two identical solutions of du.
The surface (S) cuts the tangent plane T with one line that passes through the point P (parabolic point);
$(M^2 - LN) < 0$: no solution of du.
The surface (S) does not cut the tangent plane T except at the point P (elliptic point).

In another way, the second fundamental form II can be derived by the change rate of the differential of the arc length ds when the surface (S) moves along the unit normal vector **n** with a parameterized variable α according to Chase (2012).

The location vector of the point P can be written in the parameterized variable α.

$$\mathbf{R}_u(u, v, \alpha) = \mathbf{r}(u, v) - \alpha \mathbf{n}(u, v) \tag{3.38}$$

The second fundamental form II can be calculated at $\alpha = 0$ using Eq. (3.27a):

$$\begin{aligned} \mathrm{II} &= \mathrm{d}s \cdot \frac{\partial(\mathrm{d}s)}{\partial \alpha}\Big|_{\alpha=0} = \frac{1}{2}\frac{\partial(\mathrm{d}s^2)}{\partial \alpha}\Big|_{\alpha=0} = \frac{1}{2}\frac{\partial I}{\partial \alpha}\Big|_{\alpha=0} \\ &= \frac{1}{2}\frac{\partial}{\partial \alpha}(E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2)\,|_{\alpha=0} \\ &= \frac{1}{2}\frac{\partial E}{\partial \alpha}|_{\alpha=0}\mathrm{d}u^2 + \frac{\partial F}{\partial \alpha}|_{\alpha=0}\mathrm{d}u\mathrm{d}v + \frac{1}{2}\frac{\partial G}{\partial \alpha}|_{\alpha=0}\mathrm{d}v^2 \end{aligned} \tag{3.39}$$

in which the first term in the RHS of Eq. (3.39) can be calculated as

$$\begin{aligned} E &= \mathbf{R}_u(u, v, \alpha) \cdot \mathbf{R}_u(u, v, \alpha) \\ &= (\mathbf{r}_u - \alpha\,\mathbf{n}_u) \cdot (\mathbf{r}_u - \alpha\,\mathbf{n}_u) \\ &= \mathbf{n}_u^2\alpha^2 - 2\mathbf{r}_u\mathbf{n}_u\alpha + \mathbf{r}_u^2 \end{aligned} \tag{3.40}$$

Thus,

$$\frac{1}{2}\frac{\partial E}{\partial \alpha}\,|_{\alpha=0} = (\mathbf{n}_u^2\alpha - \mathbf{r}_u\mathbf{n}_u)\,|_{\alpha=0} = -\mathbf{r}_u\mathbf{n}_u \tag{3.41}$$

Further calculations deliver the second and third terms in the RHS of Eq. (3.39):

$$\frac{\partial F}{\partial \alpha}\,|_{\alpha=0} = -(\mathbf{r}_u\mathbf{n}_v + \mathbf{r}_v\mathbf{n}_u); \tag{3.42}$$

$$\frac{1}{2}\frac{\partial G}{\partial \alpha}\,|_{\alpha=0} = -\mathbf{r}_v\mathbf{n}_v \tag{3.43}$$

Using the orthogonality of $\mathbf{r}_u$ and $\mathbf{n}$, one obtains

$$\frac{\partial(\mathbf{r}_u \cdot \mathbf{n})}{\partial u} = \mathbf{r}_{uu}\mathbf{n} + \mathbf{r}_u\mathbf{n}_u = 0 \Rightarrow \mathbf{r}_{uu}\mathbf{n} = -\mathbf{r}_u\mathbf{n}_u \tag{3.44}$$

Similarly, one obtains using the orthogonality of $\mathbf{r}_u$ and $\mathbf{n}$, and $\mathbf{r}_v$ and $\mathbf{n}$

$$\begin{aligned} &\frac{\partial(\mathbf{r}_u \cdot \mathbf{n})}{\partial v} = \frac{\partial(\mathbf{r}_v \cdot \mathbf{n})}{\partial u} = 0 \Rightarrow 2\mathbf{r}_{uv}\mathbf{n} = -(\mathbf{r}_u\mathbf{n}_v + \mathbf{r}_v\mathbf{n}_u); \\ &\frac{\partial(\mathbf{r}_v \cdot \mathbf{n})}{\partial v} = 0 \Rightarrow \mathbf{r}_{vv}\mathbf{n} = -\mathbf{r}_v\mathbf{n}_v \end{aligned} \tag{3.45}$$

Substituting Eqs. (3.41)–(3.45) into Eq. (3.39), the second fundamental form II can be written as

$$
\begin{aligned}
\mathrm{II} &= \frac{1}{2}\frac{\partial E}{\partial \alpha}\Big|_{\alpha=0} \mathrm{d}u^2 + \frac{\partial F}{\partial \alpha}\Big|_{\alpha=0} \mathrm{d}u\mathrm{d}v + \frac{1}{2}\frac{\partial G}{\partial \alpha}\Big|_{\alpha=0} \mathrm{d}v^2 \\
&= -\mathbf{r}_u\mathbf{n}_u \mathrm{d}u^2 - (\mathbf{r}_u\mathbf{n}_v + \mathbf{r}_v\mathbf{n}_u)\mathrm{d}u\mathrm{d}v - \mathbf{r}_v\mathbf{n}_v \mathrm{d}v^2 \\
&= \mathbf{r}_{uu}\mathbf{n}\mathrm{d}u^2 + 2\mathbf{r}_{uv}\mathbf{n}\mathrm{d}u\mathrm{d}v + \mathbf{r}_{vv}\mathbf{n}\mathrm{d}v^2 \\
&\equiv L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2
\end{aligned} \tag{3.46}
$$

where L, M, and N are the components of the Hessian tensor

$$
\mathbf{H} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu}\mathbf{n} & \mathbf{r}_{uv}\mathbf{n} \\ \mathbf{r}_{uv}\mathbf{n} & \mathbf{r}_{vv}\mathbf{n} \end{bmatrix} \tag{3.47}
$$

3.6 Gaussian and Mean Curvatures

The Gaussian and mean curvatures are based on the principal normal curvatures κ_1 and κ_2 in the directions $\mathbf{t}_1$ and $\mathbf{t}_2$ of the surface (S) at a given point P, respectively, as shown in Fig. 3.4. The unit tangent vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ and the unit normal vector $\mathbf{n}$ at the point P generate two principal curvature planes that are perpendicular to each other. The normal curvature κ_1 of the surface (S) in the principal direction $\mathbf{t}_1$ at the point P is defined as the maximum normal curvature in the curvature plane P_1; the normal curvature κ_2 in the principal direction $\mathbf{t}_2$ is the minimum normal curvature in the curvature plane P_2.

The maximum and minimum normal curvatures κ_1 and κ_2 of the surface (S) at the point P are the eigenvalues of the corresponding eigenvectors $\mathbf{t}_1$ and $\mathbf{t}_2$ (Bär 2001; Chase 2012; Kühnel 2013; Lang 2001). These eigenvalues are given from the characteristic equation that can be derived from the first and second fundamental forms in Eqs. (3.27a, 3.27b and 3.46).

The Gaussian curvature of the surface (S) at the point P is defined by

$$
K = \kappa_1 \kappa_2 \tag{3.48}
$$

The mean curvature of the surface (S) at the point P is defined by

$$
H = \frac{1}{2}(\kappa_1 + \kappa_2) \tag{3.49}
$$

The covariant metric tensor related to the first fundamental form can be written as

$$
\begin{aligned}
I &= E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2; \\
\mathbf{M} &= (g_{ij}) \equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u\mathbf{r}_u & \mathbf{r}_u\mathbf{r}_v \\ \mathbf{r}_v\mathbf{r}_u & \mathbf{r}_v\mathbf{r}_v \end{bmatrix}
\end{aligned} \tag{3.50}
$$

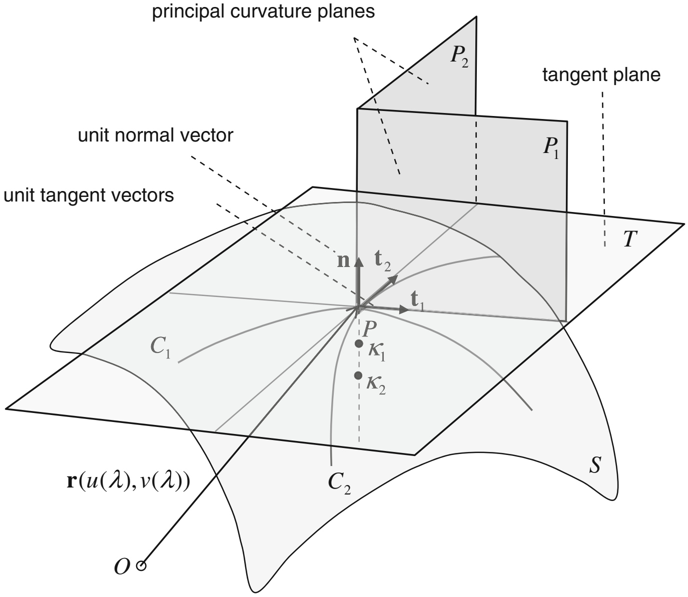

Fig. 3.4 Gaussian and mean curvatures of the surface S

The Hessian tensor related to the second fundamental form can be written as

$$II = L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2;$$
$$\mathbf{H} = (h_{ij}) \equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu}\mathbf{n} & \mathbf{r}_{uv}\mathbf{n} \\ \mathbf{r}_{uv}\mathbf{n} & \mathbf{r}_{vv}\mathbf{n} \end{bmatrix} \tag{3.51}$$

The characteristic equation of the principal curvatures results in

$$\det(\mathbf{H} - \kappa\mathbf{M}) = \begin{vmatrix} (L - \kappa E) & (M - \kappa F) \\ (M - \kappa F) & (N - \kappa G) \end{vmatrix} = 0 \tag{3.52}$$

Therefore,

$$(L - \kappa E) \cdot (N - \kappa G) - (M - \kappa F)^2 = 0 \Leftrightarrow$$
$$(EG - F^2)\kappa^2 + (EN - 2MF + LG)\kappa + (LN - M^2) = 0 \tag{3.53}$$

The Gaussian curvature results from Eq. (3.53)

$$K = \kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2} = \frac{\det(h_{ij})}{\det(g_{ij})} \tag{3.54}$$

Note that the Gaussian curvature K at a point in the surface is the product of two principal curvatures at this point. According to Gauss's Theorema Egregium (remarkable theorem) in [Bär (2001); Chase (2012); Kühnel (2013); Lang (2001)], the Gaussian curvature depends only on the first fundamental form I.

Similarly, the mean curvature results in

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = -\frac{(EN - 2MF + LG)}{2(EG - F^2)} \tag{3.55}$$

The maximum and minimum principal curvatures κ_1 and κ_2 of the surface S at the point P result from Eqs. (3.54) and (3.55).

$$\begin{cases} \kappa_1 = \kappa_{\max} = H + \sqrt{H^2 - K} \\ \kappa_2 = \kappa_{\min} = H - \sqrt{H^2 - K} \end{cases} \tag{3.56}$$

In the following section, the Gaussian and mean curvatures of the rotational paraboloid surface (S) in $\mathbf{R}^3$ are computed as follows:

$$\begin{aligned} (S) &: z = x^2 + y^2; \\ h(t) &= \sqrt{t}; \quad t > 0 \end{aligned} \tag{3.57}$$

The location vector of the curvilinear surface (S) can be written as

$$\mathbf{r}(t, h) = \begin{pmatrix} h(t)\cos\varphi \\ h(t)\sin\varphi \\ h^2(t) \end{pmatrix} \tag{3.58}$$

The bases of the curvilinear coordinate (t, φ) result from Eq. (3.58) in

$$\mathbf{r}_t = \frac{\partial \mathbf{r}}{\partial t} = \begin{pmatrix} \dot{h}(t)\cos\varphi \\ \dot{h}(t)\sin\varphi \\ 2h\dot{h}(t) \end{pmatrix}; \quad \mathbf{r}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi} = \begin{pmatrix} -h\sin\varphi \\ h\cos\varphi \\ 0 \end{pmatrix} \tag{3.59}$$

Thus, the covariant metric tensor can be further computed as

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} \mathbf{r}_t\mathbf{r}_t & \mathbf{r}_t\mathbf{r}_\varphi \\ \mathbf{r}_\varphi\mathbf{r}_t & \mathbf{r}_\varphi\mathbf{r}_\varphi \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \\ &= \begin{bmatrix} \dot{h}^2(1 + 4h^2) & 0 \\ 0 & h^2 \end{bmatrix} \end{aligned} \tag{3.60}$$

The unit normal vector can be calculated as

$$
\begin{aligned}
\mathbf{n} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\sqrt{EG - F^2}} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dot{h}\cos\varphi & \dot{h}\sin\varphi & 2h\dot{h} \\ -h\sin\varphi & h\cos\varphi & 0 \end{vmatrix} \\
&= \frac{1}{h\dot{h}\sqrt{1+4h^2}} \begin{pmatrix} -2\dot{h}h^2\cos\varphi \\ -2\dot{h}h^2\sin\varphi \\ \dot{h}h \end{pmatrix} = \frac{-1}{\sqrt{1+4h^2}} \begin{pmatrix} 2h\cos\varphi \\ 2h\sin\varphi \\ -1 \end{pmatrix}
\end{aligned}
\tag{3.61}
$$

The components of the Hessian tensor are calculated by differentiating Eq. (3.59) with respect to t and φ.

$$
\begin{aligned}
&\mathbf{r}_{tt} = \frac{\partial^2 \mathbf{r}}{\partial t^2} = \begin{pmatrix} \ddot{h}\cos\varphi \\ \ddot{h}\sin\varphi \\ 2\ddot{h}h + 2\dot{h}^2 \end{pmatrix}; \quad \mathbf{r}_{t\varphi} = \frac{\partial^2 \mathbf{r}}{\partial t \partial \varphi} = \begin{pmatrix} -\dot{h}\sin\varphi \\ \dot{h}\cos\varphi \\ 0 \end{pmatrix}; \\
&\mathbf{r}_{\varphi\varphi} = \frac{\partial^2 \mathbf{r}}{\partial \varphi^2} = \begin{pmatrix} -h\cos\varphi \\ -h\sin\varphi \\ 0 \end{pmatrix}
\end{aligned}
\tag{3.62}
$$

The Hessian tensor results from Eqs. (3.61) and (3.62) in

$$
\begin{aligned}
\mathbf{H} &= \begin{bmatrix} \mathbf{r}_{tt}\mathbf{n} & \mathbf{r}_{t\varphi}\mathbf{n} \\ \mathbf{r}_{\varphi t}\mathbf{n} & \mathbf{r}_{\varphi\varphi}\mathbf{n} \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} \\
&= \frac{2}{\sqrt{1+4h^2}} \begin{pmatrix} \dot{h}^2 & 0 \\ 0 & h^2 \end{pmatrix}
\end{aligned}
\tag{3.63}
$$

Therefore, the Gaussian curvature can be calculated from Eqs. (3.60) and (3.63).

$$
\begin{aligned}
K = \kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2} &= \frac{\left(\frac{2}{\sqrt{1+4h^2}}\right)^2 \dot{h}^2 h^2}{\dot{h}^2 h^2 (1 + 4h^2)} \\
&= \frac{4}{(1+4h^2)^2} = \frac{4}{(1+4t)^2} > 0
\end{aligned}
\tag{3.64}
$$

In the case of $(M^2 - LN) < 0$, no solution of du exists. Thus, the surface (S) does not cut the tangent plane T except at the point P that is called the elliptic point.

Analogously, the mean curvature results from Eqs. (3.60) and (3.63) in

$$
\begin{aligned}
H = \frac{1}{2}(\kappa_1 + \kappa_2) &= \frac{EN - 2MF + LG}{2(EG - F^2)} \\
&= \frac{\left(\dfrac{2\dot{h}^2 h^2}{\sqrt{1+4h^2}}\right) \cdot \left[1 + (1 + 4h^2)\right]}{2\dot{h}^2 h^2 (1 + 4h^2)} = \frac{2(1+2t)}{(1+4t)^{\frac{3}{2}}}
\end{aligned}
\tag{3.65}
$$

in which $\mathrm{h}^2 = t$.

3.7 Riemann Curvature

The Riemann curvature (also Riemann curvature tensor) is closely related to the Gaussian curvature of the surface in differential geometry (Bär 2001; Chase 2012; Klingbeil 1966). At first, let us look into the second covariant derivative of an arbitrary first-order tensor of which the first covariant derivative with respect to u^j has been derived in Eq. (2.208).

$$T_i\big|_j = T_{i,j} - \Gamma_{ij}^k T_k \tag{3.66}$$

Obviously, the covariant derivative $T_i|_j$ is a second-order tensor component.

Differentiating $T_i|_j$ with respect to u^k, the first covariant derivative of the second-order tensor (component) $T_i|_j$ is the second covariant derivative of an arbitrary first-order tensor (component) T_i. This second covariant derivative has been given from Eq. (2.211a) (Klingbeil 1966).

$$
\begin{aligned}
T_i\big|_{jk} &\equiv (T_i\big|_j)\big|_k \\
&= (T_i\big|_j)_{,k} - \Gamma_{ik}^m T_m\big|_j - \Gamma_{jk}^m T_i\big|_m \\
&= T_i\big|_{j,k} - \Gamma_{ik}^m T_m\big|_j - \Gamma_{jk}^m T_i\big|_m
\end{aligned}
\tag{3.67}
$$

Equation (3.66) delivers the relations of

$$T_i\big|_{j,k} = T_{i,jk} - (\Gamma_{ij,k}^m T_m + \Gamma_{ij}^m T_{m,k}) \tag{3.68a}$$

$$\Gamma_{ik}^m T_m\big|_j = \Gamma_{ik}^m (T_{m,j} - \Gamma_{mj}^n T_n) \tag{3.68b}$$

$$\Gamma_{jk}^m T_i\big|_m = \Gamma_{jk}^m (T_{i,m} - \Gamma_{im}^n T_n) \tag{3.68c}$$

Inserting Eqs. (3.68a), (3.68b), and (3.68c) into Eq. (3.67), one obtains the second covariant derivative of T_i.

$$\begin{aligned} T_i\big|_{jk} &= T_i\big|_{j,k} - \Gamma_{ik}^m T_m\big|_j - \Gamma_{jk}^m T_i\big|_m \\ &= T_{i,jk} - (\Gamma_{ij,k}^m T_m + \Gamma_{ij}^m T_{m,k}) \\ &\quad - \Gamma_{ik}^m (T_{m,j} - \Gamma_{mj}^n T_n) - \Gamma_{jk}^m (T_{i,m} - \Gamma_{im}^n T_n) \\ &= T_{i,jk} - \Gamma_{ij,k}^m T_m - \Gamma_{ij}^m T_{m,k} \\ &\quad - \Gamma_{ik}^m T_{m,j} + \Gamma_{ik}^m \Gamma_{mj}^n T_n - \Gamma_{jk}^m T_{i,m} + \Gamma_{jk}^m \Gamma_{im}^n T_n \end{aligned} \tag{3.69}$$

where the second partial derivative of T_i is symmetric with respect to j and k:

$$T_{i,jk} \equiv \frac{\partial^2 T_i}{\partial u^j \partial u^k} = \frac{\partial^2 T_i}{\partial u^k \partial u^j} \equiv T_{i,kj} \tag{3.70}$$

Interchanging the indices j with k in Eq. (3.69), one obtains

$$\begin{aligned} T_i\big|_{kj} &= T_{i,kj} - \Gamma_{ik,j}^m T_m - \Gamma_{ik}^m T_{m,j} \\ &\quad - \Gamma_{ij}^m T_{m,k} + \Gamma_{ij}^m \Gamma_{mk}^n T_n - \Gamma_{kj}^m T_{i,m} + \Gamma_{kj}^m \Gamma_{im}^n T_n \end{aligned} \tag{3.71}$$

Using the symmetry properties given in Eq. (3.70), Eq. (3.71) can be rewritten as

$$\begin{aligned} T_i\big|_{kj} &= T_{i,jk} - \Gamma_{ik,j}^m T_m - \Gamma_{ik}^m T_{m,j} \\ &\quad - \Gamma_{ij}^m T_{m,k} + \Gamma_{ij}^m \Gamma_{mk}^n T_n - \Gamma_{jk}^m T_{i,m} + \Gamma_{jk}^m \Gamma_{im}^n T_n \end{aligned} \tag{3.72}$$

In a flat space, the second covariant derivatives in Eqs. (3.69) and (3.72) are identical. On the contrary, they are not equal in a curved space because of its surface curvature. The difference of both second covariant derivatives is proportional to the curvature tensor. Subtracting Eq. (3.69) from Eq. (3.72), the curvature tensor results in

$$\begin{aligned} T_i\big|_{jk} - T_i\big|_{kj} &= (\Gamma_{ik,j}^n - \Gamma_{ij,k}^n + \Gamma_{ik}^m \Gamma_{mj}^n - \Gamma_{ij}^m \Gamma_{mk}^n) T_n \\ &\equiv R_{ijk}^n T_n \end{aligned} \tag{3.73}$$

Thus, the Riemann curvature (also Riemann–Christoffel tensor) can be expressed as

$$R_{ijk}^n \equiv \Gamma_{ik,j}^n - \Gamma_{ij,k}^n + \Gamma_{ik}^m \Gamma_{mj}^n - \Gamma_{ij}^m \Gamma_{mk}^n \tag{3.74}$$

It is straightforward that the Riemann–Christoffel tensor is a fourth-order tensor with respect to the indices of i, j, k, and n. They contain $81(= 3^4)$ components in a three-dimensional space.

In Eq. (3.74), the partial derivatives of the Christoffel symbols are defined by

$$\Gamma^n_{ik,j} = \frac{\partial \Gamma^n_{ik}}{\partial u^j}; \quad \Gamma^n_{ij,k} = \frac{\partial \Gamma^n_{ij}}{\partial u^k} \tag{3.75}$$

According to Eq. (2.172), the second-kind Christoffel symbol is given

$$\Gamma^k_{ij} = \tfrac{1}{2}(g_{ip,j} + g_{jp,i} - g_{ij,p})g^{kp} \tag{3.76}$$

Therefore, the Riemann curvature tensor in Eq. (3.74) only depends on the covariant and contravariant metric coefficients of the metric tensor **M**, as given in Eq. (3.28).

Furthermore, the covariant Riemann curvature tensor of fourth order is defined by the Riemann–Christoffel tensor and covariant metric coefficients.

$$R_{lijk} \equiv g_{\mathrm{ln}} R^n_{ijk} \Leftrightarrow R^n_{ijk} = g^{ln} R_{lijk} \tag{3.77}$$

For a differentiable two-dimensional manifold of the curvilinear coordinates (u,v), the Bianchi first identity gives the relation between the Riemann curvature tensors R and Gaussian curvature K, cf. Equation (3.117b).

$$R_{lijk} \equiv K \cdot (g_{lj}g_{ki} - g_{lk}g_{ji}) \tag{3.78}$$

Equation (3.78) indicates that the Gaussian curvature K of the two-dimensional surface only depends on the metric coefficients of E, F, and G. Therefore, the Gaussian curvature is only a function of the first fundamental form $\boldsymbol{I}$. This result was proved by *Gauss Theorema Egregium* (Bär 2001; Kühnel 2013; Lang 2001; Danielson 2003).

The Riemann curvature tensor has the following properties:

- First skew symmetry with respect to l and i:

$$R_{lijk} = -R_{iljk} \tag{3.79}$$

- Second skew symmetry with respect to j and k:

$$R_{lijk} = -R_{likj}; \quad R^n_{ijk} = -R^n_{ikj} \tag{3.80}$$

- Block symmetry with respect to two pairs (l, i) and (j, k):

$$R_{lijk} = R_{jkli} \tag{3.81}$$

- Cyclic property in i, j, k:

$$\begin{aligned} R_{lijk} + R_{ljki} + R_{lkij} &= 0; \\ R^n_{ijk} + R^n_{jki} + R^n_{kij} &= 0 \end{aligned} \tag{3.82}$$

Resulting from these properties, there are six components of R_{lijk} in the three-dimensional space as follows (Chase 2012):

$$R_{lijk} = R_{3131}, R_{3132}, R_{3232}, R_{1212}, R_{3112}, R_{3212} \tag{3.83}$$

In Cartesian coordinates, all second-kind Christoffel symbols equal zero according to Eq. (2.184). Therefore, the Riemann–Christoffel tensor, as given in Eq. (3.74), must be equal to zero.

$$R^n_{ijk} = \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik}\Gamma^n_{mj} - \Gamma^m_{ij}\Gamma^n_{mk} \equiv 0 \tag{3.84}$$

Therefore, the Riemann curvature tensor in Cartesian coordinates becomes

$$R_{lijk} \equiv g_{ln} R^n_{ijk} = 0 \tag{3.85}$$

3.8 Gauss–Bonnet Theorem

The Gauss–Bonnet theorem in differential geometry connects the Gaussian and geodesic curvatures of the surface to the surface topology by means of the Euler's characteristic.

Figure 3.5 displays a differentiable Riemannian surface (S) surrounded by a closed boundary curve Γ. The Gaussian curvature vector $\mathbf{K}$ is perpendicular to the manifold surface at the point P lying in the curve C and has the direction of the unit normal vector $\mathbf{n}$. The geodesic curvature vector $\mathbf{k}_g$ has the amplitude of the geodesic curvature κ_g; its direction of $(\mathbf{n} \times \mathbf{t})$ is perpendicular to the unit normal and tangent vectors in the Frenet orthonormal frame.

The Gauss–Bonnet theorem is formulated for a simple closed boundary curve Γ (Bär 2001; Chase 2012; Kühnel 2013).

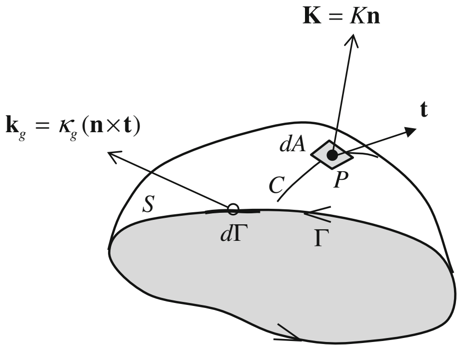

Fig. 3.5 Gaussian and geodesic curvatures for a simple closed curve Γ

$$\iint_S K \mathrm{d}A + \oint_\Gamma \kappa_g \mathrm{d}\Gamma = 2\pi \tag{3.86}$$

The compact curvilinear surface S is triangulated into a finite number of curvilinear triangles. Each triangle contains a point P on the surface. This procedure is called the surface triangulation where two neighboring curvilinear triangles have one common vertex and one common edge (see Fig. 3.6).

Therefore, the integral of the geodesic curvature over all triangles on the compact curvilinear surface S equals zero (Chase 2012).

$$\oint_\Gamma \kappa_g \mathrm{d}\Gamma = 0 \tag{3.87}$$

In the case of a compact triangulated surface, the Gauss–Bonnet theorem can be written in Euler's characteristic χ of the compact triangulated surface S_T [Bär (2001); Chase (2012); Fecko (2011)].

$$\iint_{S_T} K \mathrm{d}A = 2\pi\chi(S_T) \tag{3.88}$$

The Euler's characteristic of the compact triangulated surface can be defined by

$$\chi(S_T) \equiv V - E + F \tag{3.89}$$

in which V, E, and F are the number of vertices, edges, and faces of the considered compact triangulated surface, respectively.

Substituting Eqs. (3.8) and (3.89) into Eq. (3.88), the Gauss–Bonnet theorem can be written for a compact triangulated surface.

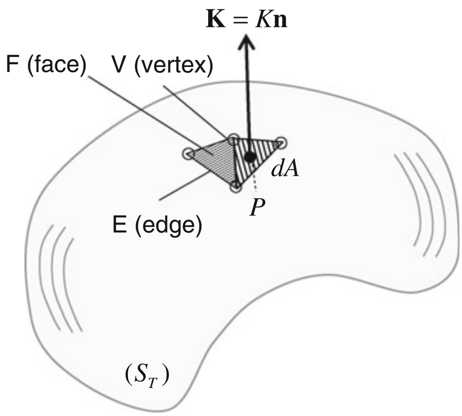

Fig. 3.6 Gaussian and geodesic curvatures for a compact triangulated surface

$$\iint_{S_T} \frac{LN - M^2}{\sqrt{EG - F^2}} \mathrm{d}u\,\mathrm{d}v = 2\pi(V - E + F) \tag{3.90}$$

3.9 Gauss Derivative Equations

Gauss derivative equations were derived from the second-kind Christoffel symbols and the basis $\mathbf{g}_3$ that denotes the normal unit vector $\mathbf{n}(=\mathbf{g}_3)$ of the curvilinear surface with the covariant bases ($\mathbf{g}_i$, $\mathbf{g}_j$) for $i, j = 1, 2$ at any point $P(u, v)$ in the surface. Note that all indices i, j, and k in the curvature surface S vary from 1 to 2.

The partial derivatives of the covariant basis $\mathbf{g}_i$ with respect to u^j can be written according to Eq. (2.166) as

$$\mathbf{g}_{i,j} = \Gamma_{ij}^k \mathbf{g}_k + \Gamma_{ij}^3 \mathbf{g}_3 \quad \text{for } i, j, k = 1, 2 \tag{3.91}$$

The Christoffel symbols in the normal direction $\mathbf{n}$ result from Eq. (2.164).

$$\Gamma_{ij}^3 = \mathbf{n} \cdot \mathbf{g}_{i,j} = \mathbf{g}_3 \cdot \mathbf{g}_{i,j} \tag{3.92}$$

Differentiating $\mathbf{g}_i \cdot \mathbf{g}_3$ and using the orthogonality of $\mathbf{g}_i$ and $\mathbf{g}_3$, the partial derivative of the covariant basis $\mathbf{g}_i$ with respect to u^j can be calculated as

$$\begin{aligned} (\mathbf{g}_i \cdot \mathbf{g}_3)_{,j} &= \mathbf{g}_{i,j} \cdot \mathbf{g}_3 + \mathbf{g}_i \cdot \mathbf{g}_{3,j} = 0 \\ \Rightarrow \mathbf{g}_{i,j} &= -\mathbf{g}_3(\mathbf{g}_i \cdot \mathbf{g}_{3,j}) = -(\mathbf{g}_3 \cdot \mathbf{g}_{3,j})\mathbf{g}_i \end{aligned} \tag{3.93}$$

Substituting Eq. (3.93) into Eq. (3.92), the Christoffel symbols in the normal direction $\mathbf{n}$ can be expressed as

$$
\begin{aligned}
\Gamma_{ij}^{3} &= \mathbf{g}_3 \cdot \mathbf{g}_{i,j} = -(\mathbf{g}_3 \cdot \mathbf{g}_3)\mathbf{g}_{3,j} \cdot \mathbf{g}_i \\
&= -\mathbf{g}_{3,j} \cdot \mathbf{g}_i = -\mathbf{n}_j \cdot \mathbf{g}_i \\
&\equiv h_{ij} = h_{ji} = \Gamma_{ji}^{3} \quad \text{for } i,j = 1,2
\end{aligned}
\tag{3.94a}
$$

Thus,

$$
\mathbf{n}_j = -h_{ij}\mathbf{g}^i \quad \text{for } i,j = 1,2 \tag{3.94b}
$$

where h_{ij} is the symmetric covariant components of the Hessian tensor **H**, as given in Eq. (3.104b).

Inserting Eq. (3.94a) into Eq. (3.91), the covariant derivative of the basis $\mathbf{g}_i$ with respect to u^j results in

$$
\begin{aligned}
\mathbf{g}_{i,j} &= \Gamma_{ij}^{k}\mathbf{g}_k + h_{ij}\mathbf{g}_3 \\
&= \Gamma_{ij}^{k}\mathbf{g}_k + h_{ij}\mathbf{n} \quad \text{for } i,j,k = 1,2
\end{aligned}
\tag{3.95}
$$

This equation is called Gauss derivative equations in which the second-kind Christoffel symbol is defined as

$$
\Gamma_{ij}^{k} = \frac{1}{2}g^{kp}\left(g_{jp,i} + g_{pi,j} - g_{ij,p}\right) \tag{3.96}
$$

3.10 Weingarten's Equations

The Weingarten's equations deal with the derivatives of the normal unit vector **n** (= $\mathbf{g}_3$) of the surface at the point $P(u, v)$ in the curvilinear coordinates $\{u^i\}$.

The covariant derivative of the normal unit vector n results from Eq. (3.91).

$$
\mathbf{n}_i \equiv \mathbf{g}_{3,i} = \Gamma_{3i}^{k}\mathbf{g}_k + \Gamma_{3i}^{3}\mathbf{g}_3 \quad \text{for } i,k = 1,2 \tag{3.97}
$$

Differentiating $\mathbf{g}_3 \cdot \mathbf{g}_3 = 1$ with respect to u^i, one obtains

$$
(\mathbf{g}_3 \cdot \mathbf{g}_3)_{,i} = 2\mathbf{g}_3 \cdot \mathbf{g}_{3,i} = 0 \Rightarrow \mathbf{g}_3 \cdot \mathbf{g}_{3,i} = 0 \tag{3.98}
$$

Using Eqs. (3.92) and (3.98), one obtains

$$
\Gamma_{ji}^{3} = \mathbf{g}_3 \cdot \mathbf{g}_{j,i} \Rightarrow \Gamma_{3i}^{3} = \mathbf{g}_3 \cdot \mathbf{g}_{3,i} = 0 \quad \text{for } i = 1,2 \tag{3.99}
$$

Inserting Eq. (3.99) into Eq. (3.97) and using Eq. (3.94b), it gives

$$\begin{aligned} \mathbf{n}_i &= \Gamma^k_{3i}\mathbf{g}_k = -h_{ik}\mathbf{g}^k = -(h_{ik}g^{kj})\mathbf{g}_j = -h^j_i\mathbf{g}_j \\ \Rightarrow \frac{\partial \mathbf{n}}{\partial u^i} &\equiv \mathbf{n}_i = -h^j_i\mathbf{g}_j \quad \text{for } i,j = 1,2 \end{aligned} \tag{3.100}$$

The mixed components h^j_i are calculated from the Hessian tensor **H** and metric tensor **M**, as shown in Eq. (3.105).

$$(h^j_i) = (h_{ik}g^{kj}) = \mathbf{H}\mathbf{M}^{-1} \tag{3.101}$$

Using Eq. (3.101) for i, $j = u$, v, the Weingarten's equations (3.100) can be written as

$$\begin{aligned} \mathbf{n}_u &= \left(\frac{FM - LG}{EG - F^2}\right)\mathbf{r}_u + \left(\frac{FL - EM}{EG - F^2}\right)\mathbf{r}_v \\ \Leftrightarrow \frac{\partial \mathbf{n}}{\partial u} &= \left(\frac{FM - LG}{EG - F^2}\right)\frac{\partial \mathbf{r}}{\partial u} + \left(\frac{FL - EM}{EG - F^2}\right)\frac{\partial \mathbf{r}}{\partial v} \end{aligned} \tag{3.102}$$

$$\begin{aligned} \mathbf{n}_v &= \left(\frac{FN - GM}{EG - F^2}\right)\mathbf{r}_u + \left(\frac{FM - EN}{EG - F^2}\right)\mathbf{r}_v \\ \Leftrightarrow \frac{\partial \mathbf{n}}{\partial v} &= \left(\frac{FN - GM}{EG - F^2}\right)\frac{\partial \mathbf{r}}{\partial u} + \left(\frac{FM - EN}{EG - F^2}\right)\frac{\partial \mathbf{r}}{\partial v} \end{aligned} \tag{3.103}$$

where the covariant metric and Hessian tensors result from the coefficients of the first and second fundamental forms I and II.

$$\begin{aligned} \mathbf{M} = (g_{ij}) &\equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u\mathbf{r}_u & \mathbf{r}_u\mathbf{r}_v \\ \mathbf{r}_v\mathbf{r}_u & \mathbf{r}_v\mathbf{r}_v \end{bmatrix} \\ \Rightarrow \mathbf{M}^{-1} = (g^{ij}) &= \frac{1}{EG - F^2}\begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \end{aligned} \tag{3.104a}$$

$$\mathbf{H} = (h_{ij}) \equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu}\mathbf{n} & \mathbf{r}_{uv}\mathbf{n} \\ \mathbf{r}_{uv}\mathbf{n} & \mathbf{r}_{vv}\mathbf{n} \end{bmatrix} \tag{3.104b}$$

Therefore,

$$\mathbf{H}\mathbf{M}^{-1} = (h^j_i) = \frac{-1}{EG - F^2}\begin{bmatrix} (FM - LG) & (FL - EM) \\ (FN - GM) & (FM - EN) \end{bmatrix} \tag{3.105}$$

3.11 Gauss–Codazzi Equations

The Gauss–Codazzi equations are based on the Gauss derivative and Weingarten's equations. The Gauss derivative equation (3.95) can be written as

$$\begin{aligned} \mathbf{g}_{i,j} &= \Gamma_{ij}^{k}\mathbf{g}_k + \Gamma_{ij}^{3}\mathbf{n} \\ &= \Gamma_{ij}^{k}\mathbf{g}_k + K_{ij}\mathbf{n} \quad \text{for } i,j,k = 1,2 \end{aligned} \tag{3.106}$$

in which the symmetric covariant components K_{ij} (= h_{ij}) of the surface curvature tensor **K** are given in Eq. (3.94a).

$$\begin{aligned} \Gamma_{ij}^{3} &= K_{ij} \in \mathbf{K}(\equiv \mathbf{H}) \\ &= K_{ji} = \Gamma_{ji}^{3} \end{aligned} \tag{3.107}$$

The covariant derivative of $\mathbf{g}_i$ with respect to u^j results from Eq. (3.106).

$$\begin{aligned} \mathbf{g}_i\big|_j &= \mathbf{g}_{i,j} - \Gamma_{ij}^{k}\mathbf{g}_k \\ &= K_{ij}\mathbf{n} \quad \text{for } i,j = 1,2 \end{aligned} \tag{3.108}$$

The Weingarten's equation (3.100) can also be written in the mixed components of the surface curvature tensor **K**.

$$\mathbf{n}_i \equiv \mathbf{g}_{3,i} = -K_i^j\mathbf{g}_j \quad \text{for } i,j = 1,2 \tag{3.109}$$

in which the mixed components of the surface curvature tensor **K** are defined according to Eq. (3.101) as

$$K_i^j = K_{ik}g^{kj} \in \mathbf{K}\mathbf{M}^{-1} \quad \text{for } i,j,k = 1,2 \tag{3.110}$$

Differentiating Eq. (3.108) with respect to u^k and using Eq. (3.109), the covariant second derivatives of the basis $\mathbf{g}_i$ can be calculated as

$$\mathbf{g}_i\big|_{jk} = K_{ij,k}\mathbf{n} + K_{ij}\mathbf{n}_k = K_{ij,k}\mathbf{n} - K_{ij}K_k^l\mathbf{g}_l \tag{3.111}$$

Similarly, the covariant second derivatives of the basis $\mathbf{g}_i$ with respect to u^k and u^j result from interchanging k with j in Eq. (3.111).

$$\mathbf{g}_i\big|_{kj} = K_{ik,j}\mathbf{n} + K_{ik}\mathbf{n}_j = K_{ik,j}\mathbf{n} - K_{ik}K_j^l\mathbf{g}_l \tag{3.112}$$

The Riemann curvature tensor R results from the difference of the covariant second derivatives of Eqs. (3.111) and (3.112) according to Eq. (3.73).

$$\begin{aligned} \mathbf{g}_i|_{jk} - \mathbf{g}_i|_{kj} &= R^l_{ijk}\mathbf{g}_l \\ &= (K_{ij,k} - K_{ik,j})\mathbf{n} + (K_{ik}K^l_j - K_{ij}K^l_k)\mathbf{g}_l \end{aligned} \tag{3.113}$$

Multiplying Eq. (3.113) by the normal unit vector $\mathbf{n}$ and using the orthogonality between $\mathbf{g}_l$ and $\mathbf{n}$, one obtains

$$\begin{aligned} &(K_{ij,k} - K_{ik,j})\mathbf{n}\cdot\mathbf{n} + (K_{ik}K^l_j - K_{ij}K^l_k)\mathbf{g}_l\cdot\mathbf{n} = R^l_{ijk}\mathbf{g}_l\cdot\mathbf{n} \\ &\Rightarrow\ (K_{ij,k} - K_{ik,j})\cdot 1 + (K_{ik}K^l_j - K_{ij}K^l_k)\cdot 0 = R^l_{ijk}\cdot 0 \end{aligned}$$

Thus,

$$K_{ij,k} - K_{ik,j} = 0 \quad \text{for } i,j,k = 1,2 \tag{3.114}$$

Equation (3.114) is called the *Codazzi's equation*.

Multiplying both sides of Eq. (3.113) by $\mathbf{g}_m$, using the Codazzi's equation, and employing the tensor contraction rules, one obtains

$$\begin{aligned} R^l_{ijk}\mathbf{g}_l\cdot\mathbf{g}_m &= (K_{ik}K^l_j - K_{ij}K^l_k)\mathbf{g}_l\cdot\mathbf{g}_m \\ &\Rightarrow R^m_{ijk}g_{lm} = (K_{ik}K^m_j - K_{ij}K^m_k)g_{lm} \end{aligned}$$

Thus, the Riemann curvature tensors can be calculated as

$$R_{lijk} = K_{ik}K_{lj} - K_{ij}K_{lk} \tag{3.115}$$

Equation (3.115) is called the *Gauss equation*. As a result, both Eqs. (3.114) and (3.115) are defined as the *Gauss–Codazzi equations*.

The Codazzi's equation (3.114) gives only two independent non-trivial terms (Klingbeil 1966; Danielson 2003):

$$K_{ij,k} = K_{ik,j} \Rightarrow\ (K_{11,2} = K_{12,1};\ K_{21,2} = K_{22,1}) \tag{3.116}$$

On the contrary, the Gauss equation delivers only one independent non-trivial term (Klingbeil 1966; Danielson 2003):

$$R_{1212} = K_{11}K_{22} - K^2_{12} \tag{3.117a}$$

Therefore, the Gaussian or total curvature K in Eq. (3.54) can be rewritten as

$$\begin{aligned} K &= \det\left(K^{j}_{i}\right) = (K^1_1K^2_2 - K^1_2K^2_1) \\ &= \frac{K_{11}K_{22} - K^2_{12}}{g_{11}g_{22} - g^2_{12}} = \frac{R_{1212}}{g} \end{aligned} \tag{3.117b}$$

in which

K_i^j is the mixed components of the mixed tensor $\mathbf{KM}^{-1}$, as shown in Eq. (3.105)
K_{ij} is the covariant components of the surface curvature tensor $\mathbf{K}$
R_{1212} is the covariant component of the Riemann curvature tensor
g is the determinant of the covariant metric tensor (g_{ij})

3.12 Lie Derivatives

The Lie derivatives (pronouncing/Li:/) named after the Norwegian mathematician Sophus Lie (1842–1899) are very useful geometrical tools in Lie algebras and Lie groups in differential geometry of curved manifolds. The Lie derivatives are based on vector fields that are tangent to the set of curves (also called congruence) of the curved manifold.

3.12.1 Vector Fields in Riemannian Manifold

The vector field tangent to the curve (A) parameterized by the geodesic parameter α can be written in the coordinate u^i of the N-dimensional manifold M, as displayed in Fig. 3.7. Note that all formulas in this section are expressed in the Einstein summation convention (cf. Sect. 2.2.2).

$$\begin{aligned} \mathbf{X} &\equiv \frac{\mathrm{d}}{\mathrm{d}\alpha} = \frac{\mathrm{d}u^i}{\mathrm{d}\alpha}\frac{\partial}{\partial u^i} \\ &\equiv X^i \frac{\partial}{\partial u^i} \quad \text{for } i = 1, 2, \ldots, N \end{aligned} \tag{3.118}$$

where X^i is the vector component in the coordinate u^i.

Similarly, the vector field tangent to the curve (B) parameterized by another geodesic parameter β can be written in the coordinate u^j.

$$\begin{aligned} \mathbf{Y} &\equiv \frac{\mathrm{d}}{\mathrm{d}\beta} = \frac{\mathrm{d}u^j}{\mathrm{d}\beta}\frac{\partial}{\partial u^j} \\ &\equiv Y^j \frac{\partial}{\partial u^j} \quad \text{for } j = 1, 2, \ldots, N \end{aligned} \tag{3.119}$$

where Y^j is the vector component in the coordinate u^j.

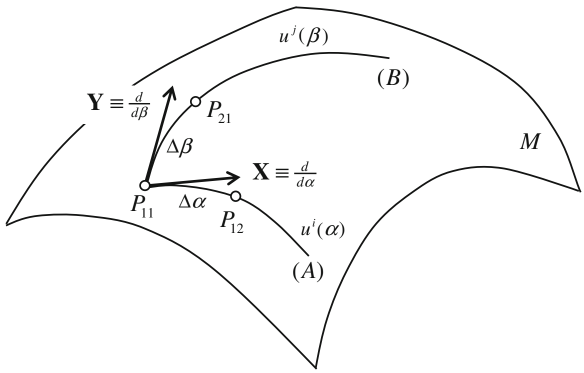

Fig. 3.7 Vector fields in the curved manifold M

3.12.2 Lie Bracket

Let $\mathbf{X}$ and $\mathbf{Y}$ be the vector fields of the congruence in the curved manifold M, and f is a mapping function of the coordinate u^i in the curve. The commutator of a vector field is called the Lie bracket and can be defined by

$$[\mathbf{X}, \mathbf{Y}] \equiv \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)) \tag{3.120}$$

The first mapping operator on the RHS of Eq. (3.120) can be calculated using the chain rule of differentiation.

$$\begin{aligned} \mathbf{X}(\mathbf{Y}(f)) &= X^j \frac{\partial}{\partial u^j}\left(Y^i \frac{\partial}{\partial u^i}\right) \\ &= X^j \frac{\partial Y^i}{\partial u^j}\frac{\partial}{\partial u^i} + X^j Y^i \frac{\partial^2}{\partial u^i \partial u^j} \end{aligned} \tag{3.121}$$

Analogously, the second mapping operator on the RHS of Eq. (3.120) results in

$$\begin{aligned} \mathbf{Y}(\mathbf{X}(f)) &= Y^j \frac{\partial}{\partial u^j}\left(X^i \frac{\partial}{\partial u^i}\right) \\ &= Y^j \frac{\partial X^i}{\partial u^j}\frac{\partial}{\partial u^i} + X^i Y^j \frac{\partial^2}{\partial u^i \partial u^j} \end{aligned} \tag{3.122}$$

Interchanging the indices i with j in the second term on the RHS of Eq. (3.122), one obtains

$$\mathbf{Y}(\mathbf{X}(f)) = Y^j \frac{\partial X^i}{\partial u^j}\frac{\partial}{\partial u^i} + X^j Y^i \frac{\partial^2}{\partial u^j \partial u^i} \tag{3.123}$$

Subtracting Eq. (3.121) from Eq. (3.123), the Lie bracket is given.

$$
\begin{aligned}
[\mathbf{X},\mathbf{Y}] &= \left(X^j\frac{\partial Y^i}{\partial u^j} - Y^j\frac{\partial X^i}{\partial u^j}\right)\frac{\partial}{\partial u^i} \\
&\equiv [\mathbf{X},\mathbf{Y}]^i\frac{\partial}{\partial u^i} \quad \text{for } i = 1,2,\ldots,N
\end{aligned}
\tag{3.124}
$$

Thus, the component i in the coordinate u^i of the Lie bracket is defined by

$$
[\mathbf{X},\mathbf{Y}]^i \equiv X^j\frac{\partial Y^i}{\partial u^j} - Y^j\frac{\partial X^i}{\partial u^j} \quad \text{for } i,j = 1,2,\ldots,N
\tag{3.125}
$$

The vectors **X** and **Y** commute if its Lie bracket equals zero.

$$
[\mathbf{X},\mathbf{Y}] = 0
\tag{3.126}
$$

According to Eq. (3.125), the Lie bracket is skew symmetric (antisymmetric).

$$
\begin{aligned}
[\mathbf{X},\mathbf{Y}] &= -[\mathbf{Y},\mathbf{X}] \\
&= -\left(Y^j\frac{\partial X^i}{\partial u^j} - X^j\frac{\partial Y^i}{\partial u^j}\right)\frac{\partial}{\partial u^i}
\end{aligned}
\tag{3.127}
$$

The Lie bracket (commutator) of the vector field (**X**, **Y**) can be expressed in another way as

$$
\begin{aligned}
[\mathbf{X},\mathbf{Y}] &= X^j\frac{\partial}{\partial u^j}\left(Y^i\frac{\partial}{\partial u^i}\right) - Y^j\frac{\partial}{\partial u^j}\left(X^i\frac{\partial}{\partial u^i}\right) \\
&= \mathbf{X}\frac{\mathrm{d}}{\mathrm{d}\beta} - \mathbf{Y}\frac{\mathrm{d}}{\mathrm{d}\alpha}
\end{aligned}
\tag{3.128}
$$

Therefore, the Lie bracket of the vector field can be written as

$$
\begin{aligned}
[\mathbf{X},\mathbf{Y}] &\equiv \left[\frac{\mathrm{d}}{\mathrm{d}\alpha},\frac{\mathrm{d}}{\mathrm{d}\beta}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\beta} - \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\mathrm{d}}{\mathrm{d}\alpha}
\end{aligned}
\tag{3.129}
$$

The Lie bracket of a vector field is generally not equal to zero in a curved manifold due to space torsions and Riemann surface curvatures that will be discussed later in Sect. 3.12.5.

The Lie bracket has some cyclic permutation properties of the vector field (**X**, **Y**, and **Z**) in the curved manifold M.

$$\begin{aligned}[\mathbf{X},[\mathbf{Y},\mathbf{Z}]] &= \mathbf{XYZ}-\mathbf{XZY}-\mathbf{YZX}+\mathbf{ZYX}\\ [\mathbf{Y},[\mathbf{Z},\mathbf{X}]] &= \mathbf{YZX}-\mathbf{YXZ}-\mathbf{ZXY}+\mathbf{XZY}\\ [\mathbf{Z},[\mathbf{X},\mathbf{Y}]] &= \mathbf{ZXY}-\mathbf{ZYZ}-\mathbf{XYZ}+\mathbf{YXZ}\end{aligned} \tag{3.130}$$

The Jacobi identity written in the Lie brackets results from substituting the properties of Eq. (3.130):

$$[\mathbf{X},[\mathbf{Y},\mathbf{Z}]]+[\mathbf{Y},[\mathbf{Z},\mathbf{X}]]+[\mathbf{Z},[\mathbf{X},\mathbf{Y}]]=0 \tag{3.131}$$

3.12.3 Lie Dragging

3.12.3.1 Lie Dragging of a Function

Let f be a mapping function $f(P_{11})$ at the point P_{11} by a geodesic parameter distance $\Delta\alpha$ into the image function $f(P_{12})$ at the point P_{12} called the image of $f(P_{11})$ on the same curve (A_i) in the manifold M (see Fig. 3.7).

If the mapping image $f(P_{12})$ at the point P_{12} equals the original function $f(P_{11})$ at the point P_{11}, the function f is called invariant under the mapping. Furthermore, the mapping function f can be defined as Lie dragged if the images of the function f are invariant for every geodesic parameter distance $\Delta\alpha$ along any congruence in the manifold M.

$$\frac{df}{d\alpha}=0 \Leftrightarrow f \text{ is Lie dragged.} \tag{3.132}$$

3.12.3.2 Lie Dragging of a Vector Field

Let $\mathbf{X}$ and $\mathbf{Y}$ be the vector fields in the curved manifold M, as shown in Fig. 3.8. The congruence consists of α and β curves with the coordinates u^i and u^j. The tangent vector $\mathbf{X}$ to the curve (A_1) at the point P_{11} is dragged to the curve (A_2) at the point P_{21} by a geodesic parameter distance $\Delta\beta$.

The image vector $\mathbf{X}^*$ of the original vector $\mathbf{X}$ is dragged by $\Delta\beta$ from (A_1) to (A_2) and tangent to the curve (A_2) at the point P_{21}. Generally, both tangent vectors $\mathbf{X}$ and $\mathbf{X}^*$ are different to each other under the Lie dragging by $\Delta\beta$. However, if they are equal for every geodesic parameter distance $\Delta\beta$, the Lie dragging is invariant. In this case, the vector field is called Lie dragged in the manifold M (Penrose 2005; Schutz 1980).

Similarly, the tangent vector $\mathbf{Y}$ to the curve (B_1) at the point P_{11} is dragged to the image vector $\mathbf{Y}^*$ tangent to the curve (B_2) at the point P_{12} by a geodesic

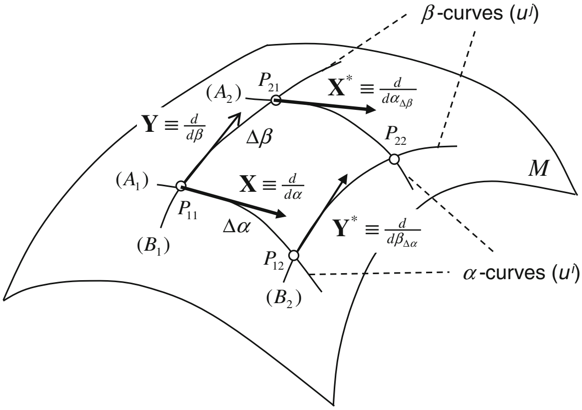

Fig. 3.8 Lie dragging a vector field in a curved manifold M

parameter distance $\Delta\alpha$. Thus, the vector field of **X** and **Y** along the congruence is generated in the manifold M. The vector field is defined as Lie dragged if its Lie bracket or the commutator given in Eq. (3.129) equals zero.

$$\begin{aligned}
[\mathbf{X},\mathbf{Y}] &\equiv \left[\frac{\mathrm{d}}{\mathrm{d}\alpha},\frac{\mathrm{d}}{\mathrm{d}\beta}\right] = \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\beta} - \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\mathrm{d}}{\mathrm{d}\alpha} = 0 \\
\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\beta} &= \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\mathrm{d}}{\mathrm{d}\alpha}
\end{aligned} \tag{3.133}$$

In this case, the vector field is commute under the Lie-dragged procedure in the manifold M. In general, the Lie bracket of a vector field is not always equal to zero due to space torsions besides the Riemann surface curvature in the curved space.

3.12.4 Lie Derivatives

The Lie derivatives of a function with respect to the vector field **X** are defined by the change rate of the function f between two different points P_{11} and P_{12} in the same curve under the Lie dragging by a geodesic parameter distance $\Delta\alpha$.

$$\begin{aligned}
£_{\mathbf{X}} f &\equiv \lim_{\Delta\alpha\to 0} \frac{f(\alpha_0+\Delta\alpha)-f(\alpha_0)}{\Delta}\alpha \\
&= \left.\frac{\mathrm{d}f(u^i)}{\mathrm{d}\alpha}\right|_{\alpha_0} = \left.\frac{\mathrm{d}u^i}{\mathrm{d}\alpha}\frac{\partial f}{\partial u^i}\right|_{\alpha_0} \\
&= X^i\frac{\partial f}{\partial u^i}
\end{aligned} \tag{3.134}$$

Thus, the Lie derivative of a function *f* with respect to the vector field **X** can be simply expressed in

$$£_{\mathbf{X}} f = \left(X^i \frac{\partial}{\partial u^i} \right) f = \mathbf{X} f \tag{3.135}$$

Analogously, the Lie derivative of a vector **Y** with respect to the vector field **X** results from the Lie bracket (Schutz 1980; Fecko 2011).

$$\begin{aligned} £_{\mathbf{X}} \mathbf{Y} &= [\mathbf{X}, \mathbf{Y}] = [\mathbf{X}, \mathbf{Y}]^i \frac{\partial}{\partial u^i} \\ &= (£_{\mathbf{X}} \mathbf{Y})^i \frac{\partial}{\partial u^i} \quad \text{for } i = 1, 2, \ldots, N \end{aligned} \tag{3.136}$$

According to Eq. (3.125), the component i in the coordinate u^i of the Lie derivative of the vector **Y** with respect to the vector field **X** can be calculated as

$$\begin{aligned} (£_{\mathbf{X}} \mathbf{Y})^i &= [\mathbf{X}, \mathbf{Y}]^i \\ &= \left(X^j \frac{\partial}{\partial u^j} \right) Y^i - \left(Y^j \frac{\partial}{\partial u^j} \right) X^i \\ &= \frac{\mathrm{d}}{\mathrm{d}\alpha} Y^i - \frac{\mathrm{d}}{\mathrm{d}\beta} X^i \quad \text{for } i = 1, 2, \ldots, N \end{aligned} \tag{3.137}$$

The Lie derivative of a vector field is skew symmetric because of the skew symmetry of the Lie bracket, as shown in Eq. (3.127).

$$£_{\mathbf{X}} \mathbf{Y} = -£_{\mathbf{Y}} \mathbf{X} \tag{3.138}$$

In the following section, the properties of the Lie derivatives are proved.

3.12.4.1 Lie Derivative of a Function Product

$$£_{\mathbf{X}} (fg) = (£_{\mathbf{X}} f) g + f (£_{\mathbf{X}} g) \tag{3.139}$$

Proof

$$\begin{aligned} £_{\mathbf{X}} (fg) &= \mathbf{X} (fg) = X^i \frac{\partial}{\partial u^i} (fg) \\ &= \left(X^i \frac{\partial f}{\partial u^i} \right) g + f \left(X^i \frac{\partial g}{\partial u^i} \right) \\ &= (£_{\mathbf{X}} f) g + f (£_{\mathbf{X}} g) \quad \text{(q.e.d.)} \end{aligned}$$

3.12.4.2 Lie Derivative of a Tensor Product

$$\pounds_{\mathbf{X}}(\mathbf{T}\otimes\mathbf{S}) = (\pounds_{\mathbf{X}}\mathbf{T})\otimes\mathbf{S} + \mathbf{T}\otimes(\pounds_{\mathbf{X}}\mathbf{S}) \tag{3.140}$$

Proof

$$\begin{aligned}
\pounds_{\mathbf{X}}(\mathbf{T}\otimes\mathbf{S}) &= \mathbf{X}(\mathbf{T}\otimes\mathbf{S}) = X^i\frac{\partial}{\partial u^i}(\mathbf{T}\otimes\mathbf{S}) \\
&= \left(X^i\frac{\partial\mathbf{T}}{\partial u^i}\right)\otimes\mathbf{S} + \mathbf{T}\otimes\left(X^i\frac{\partial\mathbf{S}}{\partial u^i}\right) \\
&= (\pounds_{\mathbf{X}}\mathbf{T})\otimes\mathbf{S} + \mathbf{T}\otimes(\pounds_{\mathbf{X}}\mathbf{S}) \quad \text{(q.e.d.)}
\end{aligned}$$

3.12.4.3 Lie Derivative of a One-Form Field Differential

$$\pounds_{\mathbf{X}}(\mathrm{d}\omega) = \mathrm{d}(\pounds_{\mathbf{X}}\omega) \tag{3.141}$$

Proof The differential of a one-form field ω consists of scalar functions and vector fields. It can be written in the coordinate u^j using the Einstein's summation convention.

$$\mathrm{d}\omega = \frac{\partial\omega}{\partial u^j}\mathrm{d}u^j \quad \text{for } j = 1, 2, \ldots, N \tag{3.142}$$

The term on the LHS of Eq. (3.141) can be calculated using Eq. (3.142).

$$\begin{aligned}
\pounds_{\mathbf{X}}(\mathrm{d}\omega) &= \pounds_{\mathbf{X}}\left(\frac{\partial\omega}{\partial u^j}\mathrm{d}u^j\right) \\
&= \pounds_{\mathbf{X}}\left(\frac{\partial\omega}{\partial u^j}\right)\mathrm{d}u^j + \frac{\partial\omega}{\partial u^j}\pounds_{\mathbf{X}}(\mathrm{d}u^j)
\end{aligned}$$

within

$$\begin{aligned}
\pounds_{\mathbf{X}}(\mathrm{d}u^j) &= \mathrm{d}(\pounds_{\mathbf{X}}u^j) = \mathrm{d}X^j \\
&= \frac{\partial X^j}{\partial u^i}\mathrm{d}u^i \quad \text{for } i = 1, 2, \ldots, N
\end{aligned}$$

Interchanging i with j in the last term of the second line on the RHS of Eq. (3.143), one obtains the term on the LHS of Eq. (3.141).

$$\begin{aligned}\pounds_{\mathbf{X}}(\mathrm{d}\omega) &= \pounds_{\mathbf{X}}\left(\frac{\partial\omega}{\partial u^j}\mathrm{d}u^j\right)\\ &= X^i\frac{\partial}{\partial u^i}\left(\frac{\partial\omega}{\partial u^j}\right)\mathrm{d}u^j + \frac{\partial\omega}{\partial u^j}\left(\frac{\partial X^j}{\partial u^i}\mathrm{d}u^i\right)\ (i \leftrightarrow j)\\ &= X^i\frac{\partial}{\partial u^i}\left(\frac{\partial\omega}{\partial u^j}\right)\mathrm{d}u^j + \frac{\partial\omega}{\partial u^i}\left(\frac{\partial X^i}{\partial u^j}\mathrm{d}u^j\right)\\ &= \left(X^i\frac{\partial^2\omega}{\partial u^i\partial u^j} + \frac{\partial\omega}{\partial u^i}\frac{\partial X^i}{\partial u^j}\right)\mathrm{d}u^j\end{aligned} \tag{3.143}$$

Next, the term on the RHS of Eq. (3.141) is computed using the chain rule of differentiation.

$$\begin{aligned}\mathrm{d}(\pounds_{\mathbf{X}}\omega) = \mathrm{d}(\mathbf{X}\omega) &= \mathrm{d}\left(X^i\frac{\partial}{\partial u^i}\omega\right)\\ &= X^i\mathrm{d}\left(\frac{\partial\omega}{\partial u^i}\right) + \frac{\partial\omega}{\partial u^i}\mathrm{d}X^i\\ &= \left(X^i\frac{\partial^2\omega}{\partial u^i\partial u^j} + \frac{\partial\omega}{\partial u^i}\frac{\partial X^i}{\partial u^j}\right)\mathrm{d}u^j\end{aligned} \tag{3.144}$$

Comparing Eqs. (3.143) to (3.144), it is proved that

$$\pounds_{\mathbf{X}}(\mathrm{d}\omega) = \mathrm{d}(\pounds_{\mathbf{X}}\omega)$$

This equation is called the Cartan's formula in the special case.
The Cartan's formula for a one-form field is given by

$$\pounds_{\mathbf{X}}\omega = i_{\mathbf{X}}\mathrm{d}\omega + \mathrm{d}(i_{\mathbf{X}}\omega)$$

The notation $i_{\mathbf{X}}\omega$ is called the interior product of ω with respect to the vector field $\mathbf{X}$ and is defined by

$$\begin{aligned}& i_{\mathbf{X}}\omega \equiv X^j i\left(\frac{\partial}{\partial u^j}\right)\omega = 0 \quad \text{for all } \omega\\ & \Rightarrow \pounds_{\mathbf{X}}\omega \equiv \mathbf{X}\omega = i_{\mathbf{X}}\mathrm{d}\omega\end{aligned}$$

Changing ω into $\mathrm{d}\omega$ in the Cartan's formula, Eq. (3.141) is proved due to

$$\mathrm{d}(\mathrm{d}\omega) = 0$$

$$\begin{aligned}\pounds_{\mathbf{X}}(\mathrm{d}\omega) &= i_{\mathbf{X}}\mathrm{d}(\mathrm{d}\omega) + \mathrm{d}(i_{\mathbf{X}}\mathrm{d}\omega)\\ &= 0 + \mathrm{d}(\pounds_{\mathbf{X}}\omega)\\ &= \mathrm{d}(\pounds_{\mathbf{X}}\omega) \quad \text{(q.e.d.)}\end{aligned}$$

3.12.4.4 Lie Derivative of a One-Form Field and Vector Product

$$\pounds_{\mathbf{X}}(\omega\mathbf{Y}) - (\pounds_{\mathbf{X}}\omega)\mathbf{Y} = \omega[\mathbf{X},\mathbf{Y}] \tag{3.145}$$

Proof

$$\begin{aligned}\pounds_{\mathbf{X}}(\omega\mathbf{Y}) &= \mathbf{X}(\omega\mathbf{Y})\\ &= X^i\frac{\partial}{\partial u^i}(\omega\mathbf{Y})\\ &= X^i\frac{\partial\omega}{\partial u^i}\mathbf{Y} + X^i\frac{\partial\mathbf{Y}}{\partial u^i}\omega\\ &= (\mathbf{X}\omega)\mathbf{Y} + (\mathbf{X}\mathbf{Y})\omega\\ &= (\pounds_{\mathbf{X}}\omega)\mathbf{Y} + (\pounds_{\mathbf{X}}\mathbf{Y})\omega\end{aligned}$$

Therefore,

$$\begin{aligned}\pounds_{\mathbf{X}}(\omega\mathbf{Y}) - (\pounds_{\mathbf{X}}\omega)\mathbf{Y} &= (\pounds_{\mathbf{X}}\mathbf{Y})\omega\\ &= \omega[\mathbf{X},\mathbf{Y}] \qquad \text{(q.e.d.)}\end{aligned}$$

3.12.4.5 Lie Derivative of a One-Form Field

$$(\pounds_{\mathbf{X}}\omega)_i = X^j\frac{\partial\omega_i}{\partial u^j} + \omega_j\frac{\partial X^j}{\partial u^i} \tag{3.146}$$

Proof From Eqs. (3.137 and 3.145), one obtains by interchanging the index i with j.

$$\begin{aligned}(\pounds_{\mathbf{X}}\omega)_i Y^i &= \pounds_{\mathbf{X}}(\omega_i Y^i) - \omega_i(\pounds_{\mathbf{X}}\mathbf{Y})^i\\ &= X^j\frac{\partial(\omega_i Y^i)}{\partial u^j} - \omega_i\left(X^j\frac{\partial Y^i}{\partial u^j} - Y^j\frac{\partial X^i}{\partial u^j}\right)\\ &= X^j\omega_i\frac{\partial Y^i}{\partial u^j} + X^jY^i\frac{\partial\omega_i}{\partial u^j} - \omega_i X^j\frac{\partial Y^i}{\partial u^j} + \omega_i Y^j\frac{\partial X^i}{\partial u^j}\\ &= X^jY^i\frac{\partial\omega_i}{\partial u^j} + \omega_i Y^j\frac{\partial X^i}{\partial u^j}\ (i \leftrightarrow j)\\ &= \left(X^j\frac{\partial\omega_i}{\partial u^j} + \omega_j\frac{\partial X^j}{\partial u^i}\right)Y^i\end{aligned}$$

Therefore,

$$(\pounds_{\mathbf{X}}\omega)_i = X^j \frac{\partial \omega_i}{\partial u^j} + \omega_j \frac{\partial X^j}{\partial u^i} \qquad \text{(q.e.d.)}$$

Multiplying Eq. (3.146) by $\mathrm{d}u^i$, one obtains interchanging i with j.

$$\begin{aligned}(\pounds_{\mathbf{X}}\omega)_i \mathrm{d}u^i &= X^j \frac{\partial \omega_i}{\partial u^j}\mathrm{d}u^i + \omega_j \frac{\partial X^j}{\partial u^i}\mathrm{d}u^i \ (i \leftrightarrow j) \\ &= (\mathbf{X}\omega_i)\mathrm{d}u^i + \omega_i \mathrm{d}X^i\end{aligned} \tag{3.147}$$

The term on the LHS of Eq. (3.147) can be written as

$$(\pounds_{\mathbf{X}}\omega)_i \mathrm{d}u^i = \pounds_{\mathbf{X}}\omega \equiv \pounds_{\mathbf{X}}(\omega_i \mathrm{d}u^i) \tag{3.148}$$

where the one-form ω can be defined by

$$\omega \equiv \omega_i \mathrm{d}u^i \tag{3.149}$$

Thus, one obtains the property of the Lie derivative of a one-form field.

$$\pounds_{\mathbf{X}}\omega = (\mathbf{X}\omega_i)\mathrm{d}u^i + \omega_i \mathrm{d}X^i \quad \text{(q.e.d.)} \tag{3.150}$$

3.12.5 Torsion and Curvature in a Distorted and Curved Manifold

The normal vector field **N** perpendicular to the surface of the manifold M is dragged in two different paths from the same point P_{11} via P_{21} to Q in the one path and via P_{12} to S in the other path, as shown in Fig. 3.9. Due to the effect of space torsions and surface curvatures, the vector field **N** does not close the connection loop at the path ends Q and S of the dragging paths. The gap of the path ends is $O(\varepsilon^2)$ in a distorted and curved manifold and is reduced to the order of $O(\varepsilon^3)$ in an only curved manifold (Penrose 2005).

The Lie derivative of the vector **Y** with respect to the vector field **X** is induced by the space torsion of the distorted manifold. As a consequence, the torsion tensor $\varepsilon^2[\mathbf{X}, \mathbf{Y}]$ generates the open connection gap QR (see Fig. 3.9). The Riemann surface curvature is to blame for the other open connection gap RS on the order of $O(\varepsilon^3)$ in the curved manifold.

Therefore, the connection loop is always closed in a torsion-free and flat space.

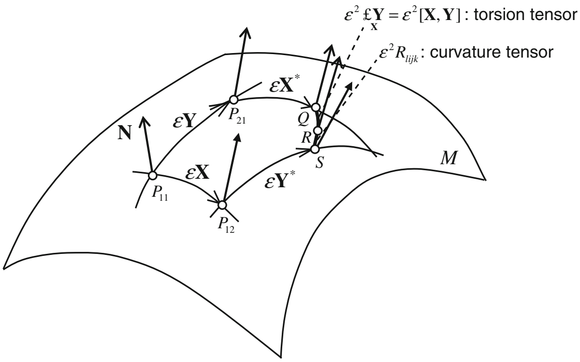

Fig. 3.9 Connection loop of vector fields in a distorted and curved manifold *M*

$$[\mathbf{Y}, \mathbf{X}] - [\mathbf{X}, \mathbf{Y}] = 0 \Leftrightarrow \pounds_{\mathbf{Y}} \mathbf{X} = \pounds_{\mathbf{X}} \mathbf{Y} \tag{3.151}$$

The curvature equation of a distorted and curved manifold can be written by means of the Lie formulations, Riemann curvature tensors, and covariant metric coefficients (Penrose 2005).

$$\begin{aligned} &[\mathbf{Y}, \mathbf{X}] - [\mathbf{X}, \mathbf{Y}] = \varepsilon^2[\mathbf{X}, \mathbf{Y}] + \varepsilon^2 R_{lijk} \\ &\Leftrightarrow \pounds_{\mathbf{Y}} \mathbf{X} - \pounds_{\mathbf{X}} \mathbf{Y} = \varepsilon^2 \pounds_{\mathbf{X}} \mathbf{Y} + \varepsilon^2 g_{ln} R^n_{ijk} \end{aligned} \tag{3.152}$$

where R_{lijk} is the Riemann curvature tensors of the curved manifold *M*.

3.12.6 Killing Vector Fields

The Killing vector field **K** is defined as a vector field in a curved manifold in which the Lie derivative of the metric tensor **g** with respect to the vector field **K** along the congruence equals zero.

$$\pounds_{\mathbf{K}} \mathbf{g} = 0 \tag{3.153}$$

Equation (3.153) shows that the metric tensor **g** is invariant in the curved manifold with respect to the Killing vector field **K**.

The covariant tensor components of the Lie derivative of the metric tensor with respect to the Killing vector field **K** can be expressed in (Schutz 1980).

$$\begin{aligned} (\pounds_{\mathbf{K}} \mathbf{g})_{ij} &= K^k \frac{\partial g_{ij}}{\partial u^k} + g_{ik} \frac{\partial K^k}{\partial u^j} + g_{kj} \frac{\partial K^k}{\partial u^i} \\ &= 0 \end{aligned} \tag{3.154}$$

Equation (3.154) can be written in one-dimensional coordinate u^k with respect to the Killing vector field **K**.

$$(\pounds_{\mathbf{K}}\,\mathbf{g})_{ij} = \frac{\partial g_{ij}}{\partial u^k} = 0 \tag{3.155}$$

Therefore, if the covariant metric coefficient is independent of any coordinate, the basis of the coordinate is a Killing vector.

As an example for the Killing vector field, the covariant metric coefficients of the spherical coordinates (r, ϕ, θ) are given in

$$\begin{cases} g_{rr} = \mathbf{g}_r \cdot \mathbf{g}_r \equiv \dfrac{\partial}{\partial r} \cdot \dfrac{\partial}{\partial r} = 1 \\ g_{\phi\phi} = \mathbf{g}_\phi \cdot \mathbf{g}_\phi \equiv \dfrac{\partial}{\partial \phi} \cdot \dfrac{\partial}{\partial \phi} = r^2 \\ g_{\theta\theta} = \mathbf{g}_\theta \cdot \mathbf{g}_\theta \equiv \dfrac{\partial}{\partial \theta} \cdot \dfrac{\partial}{\partial \theta} = r^2 \sin^2 \phi \end{cases} \tag{3.156}$$

Equation (3.156) shows that the metric coefficients are independent of the coordinates (r, ϕ, θ). Hence, the basis vectors $\mathbf{g}_r$, $\mathbf{g}_\phi$, and $\mathbf{g}_\theta$ are the Killing vectors.

3.13 Invariant Time Derivatives on Moving Surfaces

In the following section, the invariant time derivatives of tensors are applied to a surface $S(t)$ moving with a velocity vector **V** in the ambient coordinate system. For this case, the invariant time derivative of an invariant field $T(t, S)$ parameterized by the time t and moving surface S can be calculated in the surface coordinate. Generally, two coordinates of the unchanged ambient coordinate x^i with the covariant basis $\mathbf{g}_i$ and the moving surface coordinate u^α with the covariant basis $\mathbf{g}_\alpha$ are used in the moving surface $S(t)$, as shown in Fig. 3.10.

The surface S at the time t moves to the new surface position S' at the time $t + \Delta t$ at which the invariant field T(t, S) at the time t is changed into T(t + Δt, S') in a very short time interval Δt. The time-dependent surface S moves with a coordinate velocity V^i in the ambient coordinate x^i (Grinfeld 2013).

The ambient coordinate velocity of the moving surface $S(t)$ in the coordinate x^i can be defined by

$$V^i \equiv \frac{\partial x^i(t, S)}{\partial t} \tag{3.157}$$

The tangential coordinate velocity V^α results from projecting the ambient coordinate velocity V^i onto the surface along the surface coordinate u^α. To calculate the tangential coordinate velocity, the surface velocity vector **V** can be formulated in both ambient and surface coordinates using the chain rule of differentiation.

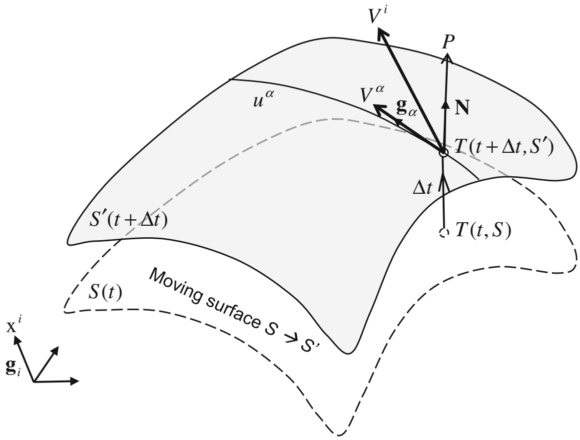

Fig. 3.10 Invariant fields on a moving surface $S(t)$

$$
\begin{aligned}
\mathbf{V} &= V^{\alpha}\mathbf{g}_{\alpha} \\
&= V^{i}\mathbf{g}_{i} = V^{i}\frac{\partial \mathbf{r}}{\partial x^{i}} \\
&= V^{i}\frac{\partial \mathbf{r}}{\partial u^{\alpha}}\frac{\partial u^{\alpha}}{\partial x^{i}} = V^{i}\frac{\partial u^{\alpha}}{\partial x^{i}}\mathbf{g}_{\alpha} \\
&\equiv (V^{i}x^{\alpha}_{,i})\mathbf{g}_{\alpha}
\end{aligned}
\tag{3.158}
$$

where the derivative $x^{\alpha}_{,i}$ is called the shift tensor between the ambient and surface coordinates.

Thus, the tangential coordinate velocity V^{α} results from the ambient coordinate velocity and shift tensor.

$$V^{\alpha} = V^{i}\frac{\partial u^{\alpha}}{\partial x^{i}} = V^{i}x^{\alpha}_{,i} \tag{3.159a}$$

Analogously, one obtains

$$V^{i} = V^{\alpha}\frac{\partial x^{i}}{\partial u^{\alpha}} = V^{\alpha}x^{i}_{,\alpha} \tag{3.159b}$$

3.13.1 Invariant Time Derivative of an Invariant Field

The invariant time derivative of an invariant field $T(t, S)$ can be defined as the time change rate of the invariant field itself and its change rates along the surface coordinates between the old and new surface positions (Grinfeld 2013).

$$\dot{\nabla} T \equiv \frac{\partial T(t,S)}{\partial t} - V^{\alpha} \nabla_{\alpha} T \tag{3.160}$$

At first, the covariant surface derivative of a first-order tensor **T** can be written as

$$\begin{aligned} \nabla_{\alpha} \mathbf{T} &= \frac{\partial \mathbf{T}}{\partial u^{\alpha}} = \frac{\partial (T^i \mathbf{g}_i)}{\partial u^{\alpha}} \\ &= \frac{\partial T^i}{\partial u^{\alpha}} \mathbf{g}_{\mathrm{i}} + T^i \frac{\partial \mathbf{g}_i}{\partial u^{\alpha}} \\ &= \frac{\partial T^i}{\partial u^{\alpha}} \mathbf{g}_{\mathrm{i}} + T^i \frac{\partial \mathbf{g}_i}{\partial x^j} \frac{\partial x^j}{\partial u^{\alpha}} \end{aligned} \tag{3.161a}$$

Using Eq. (2.158), the covariant derivative of the basis $\mathbf{g}_i$ results in

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \Gamma_{ij}^k \mathbf{g}_k \tag{3.161b}$$

Inserting Eq. (3.161b) into Eq. (3.161a), one obtains

$$\begin{aligned} \nabla_{\alpha} \mathbf{T} &= \frac{\partial T^i}{\partial u^{\alpha}} \mathbf{g}_i + \frac{\partial x^j}{\partial u^{\alpha}} \Gamma_{ij}^k T^i \mathbf{g}_k \\ &= \left(\frac{\partial T^k}{\partial u^{\alpha}} + \frac{\partial x^j}{\partial u^{\alpha}} \Gamma_{ij}^k T^i \right) \mathbf{g}_k \\ &\equiv (\nabla_{\alpha} T^k) \mathbf{g}_k \end{aligned} \tag{3.161c}$$

Thus,

$$\begin{aligned} \nabla_{\alpha} T^k &= \frac{\partial T^k}{\partial u^{\alpha}} + \frac{\partial x^j}{\partial u^{\alpha}} \Gamma_{ij}^k T^i \\ \Leftrightarrow \nabla_{\alpha} T^i &= \frac{\partial T^i}{\partial u^{\alpha}} + \frac{\partial x^j}{\partial u^{\alpha}} \Gamma_{jm}^i T^m \end{aligned} \tag{3.161d}$$

The covariant surface derivative of a contravariant first-order tensor results in using Eq. (3.161d) and the chain rule of coordinates.

$$\begin{aligned} \nabla_{\alpha} T^i \equiv T^i\big|_{\alpha} &= \frac{\partial T^i}{\partial u^{\alpha}} + \frac{\partial x^j}{\partial u^{\alpha}} \Gamma_{jm}^i T^m \\ &= \frac{\partial T^i}{\partial x^j} \frac{\partial x^j}{\partial u^{\alpha}} + \frac{\partial x^j}{\partial u^{\alpha}} \Gamma_{jm}^i T^m \\ &= \frac{\partial x^j}{\partial u^{\alpha}} \left(\frac{\partial T^i}{\partial x^j} + \Gamma_{jm}^i T^m \right) \\ &= x^j_{,\alpha} T^i\big|_j \equiv x^j_{,\alpha} \nabla_j T^i \end{aligned} \tag{3.161e}$$

Analogously, the covariant surface derivative of a mixed second-order tensor using Eq. (2.211a) and the chain rule of coordinates results in

$$\begin{aligned}\nabla_\alpha T^i_j \equiv T^i_j\Big|_\alpha &= \frac{\partial T^i_j}{\partial u^\alpha} + \frac{\partial x^k}{\partial u^\alpha}\left(\Gamma^i_{km}T^m_j - \Gamma^n_{kj}T^i_n\right) \\ &= \frac{\partial T^i_j}{\partial x^k}\frac{\partial x^k}{\partial u^\alpha} + \frac{\partial x^k}{\partial u^\alpha}\left(\Gamma^i_{km}T^m_j - \Gamma^n_{kj}T^i_n\right) \\ &= \frac{\partial x^k}{\partial u^\alpha}\left(\frac{\partial T^i_j}{\partial x^k} + \Gamma^i_{km}T^m_j - \Gamma^n_{kj}T^i_n\right) \\ &= x^k_{,\alpha}T^i_j\Big|_k \equiv x^k_{,\alpha}\nabla_k T^i_j\end{aligned} \tag{3.161f}$$

The ambient coordinate is dependent on time and the surface coordinate of the moving surface S. Thus, the invariant field T can be expressed as

$$T(t,S) = T(t,\, x(t,S)) \tag{3.162}$$

Using the Taylor series and chain rule, the partial time derivative of T with two independent variables of t and S can be calculated as

$$\begin{aligned}\frac{\partial T(t,S)}{\partial t} &= \frac{\partial T(t,x)}{\partial t} + \frac{\partial T(t,x)}{\partial x^i}\cdot\frac{\partial x^i(t,S)}{\partial t} \\ &= \frac{\partial T(t,x)}{\partial t} + (\nabla_i T)V^i\end{aligned} \tag{3.163}$$

because

$$\nabla_i T = \frac{\partial T(t,x)}{\partial x^i}; \qquad V^i = \frac{\partial x^i(t,S)}{\partial t} \tag{3.164}$$

The invariant time derivative in Eq. (3.160) can be rewritten using Eqs. (3.161e) and (3.163).

$$\begin{aligned}\dot{\nabla} T &\equiv \frac{\partial T(t,S)}{\partial t} - V^\alpha \nabla_\alpha T \\ &= \frac{\partial T(t,x)}{\partial t} + V^i\nabla_i T - V^\alpha x^k_{,\alpha}\nabla_k T \\ &= \frac{\partial T(t,x)}{\partial t} + V^i\nabla_i T - V^\alpha x^i_{,\alpha}\nabla_i T \\ &= \frac{\partial T(t,x)}{\partial t} + (V^i - x^i_{,\alpha}V^\alpha)\nabla_i T\end{aligned} \tag{3.165a}$$

The second term on the RHS in Eq. (3.165a) can be further calculated using Eq. (3.159a).

$$\begin{aligned} V^i - x^i_{,\alpha} V^\alpha &= V^i - \frac{\partial x^i}{\partial u^\alpha}\left(\frac{\partial u^\alpha}{\partial x^j} V^j\right) \\ &= V^i - x^i_{,\alpha} x^\alpha_{;j} V^j \\ &= V^j(\delta^i_j - x^i_{,\alpha} x^\alpha_{;j}) \end{aligned} \tag{3.165b}$$

The useful relation between the contra- and covariant normal vector components and shift tensors of coordinates is derived by (Grinfeld 2013)

$$N^i N_j + x^i_{,\alpha} x^\alpha_{;j} = \delta^i_j \tag{3.165c}$$

in which δ^i_j is the Kronecker delta.

Substituting Eq. (3.165c) into Eq. (3.165b), the invariant time derivative of T given in Eq. (3.165a) can be rewritten as

$$\begin{aligned} \dot{\nabla} T &= \frac{\partial T(t,x)}{\partial t} + (V^j N_j) N^i \nabla_i T \\ &= \frac{\partial T(t,x)}{\partial t} + P N^i \nabla_i T \end{aligned} \tag{3.165d}$$

where P is the normal velocity at a given point on the moving surface S, as displayed in Fig. 3.10. In fact, the normal velocity is the projection of the ambient coordinate velocity V^i on the surface normal N_i.

$$\begin{aligned} &P = V^i N_i \Rightarrow \\ &\mathbf{P} = P\mathbf{N} \\ &\quad = (V^i N_i)\mathbf{N} = V^i N_i N^j \mathbf{g}_j \end{aligned} \tag{3.165e}$$

3.13.2 Invariant Time Derivative of Tensors

Analogously, the invariant time derivative of tensors can be derived from the invariant field. The contravariant tensor can be written in the covariant basis $\mathbf{g}_i$.

$$\mathbf{T} = T^i \mathbf{g}_i \tag{3.166}$$

The invariant time derivative of the tensor $\mathbf{T}$ can be expressed on the moving surface S according to Eq. (3.160) (Grinfeld 2013).

$$\dot{\nabla}\mathbf{T} = \frac{\partial \mathbf{T}(t,S)}{\partial t} - V^\alpha \nabla_\alpha \mathbf{T} \tag{3.167}$$

Substituting Eq. (3.166) into Eq. (3.167), one obtains

$$\begin{aligned}\dot{\nabla}\mathbf{T} &= \frac{\partial(T^i\mathbf{g}_i)}{\partial t} - V^\alpha\nabla_\alpha(T^i\mathbf{g}_i)\\ &= \frac{\partial T^i}{\partial t}\mathbf{g}_i + T^i\frac{\partial \mathbf{g}_i}{\partial t} - V^\alpha(\nabla_\alpha T^i)\mathbf{g}_i\end{aligned} \tag{3.168}$$

Using Eqs. (2.158) and (3.157), the time derivative of the coordinate basis $\mathbf{g}_i$ in Eq. (3.168) can be calculated.

$$\begin{aligned}\dot{\mathbf{g}}_i &\equiv \frac{\partial \mathbf{g}_i}{\partial t} = \frac{\partial \mathbf{g}_i}{\partial x^j}\frac{\partial x^j}{\partial t} = \mathbf{g}_{i,j}\frac{\partial x^j}{\partial t}\\ &= \Gamma^k_{ij}\frac{\partial x^j}{\partial t}\mathbf{g}_k = \Gamma^k_{ij}V^j\mathbf{g}_k\end{aligned} \tag{3.169}$$

Therefore, the invariant time derivative of the tensor $\mathbf{T}$ can be rewritten as

$$\begin{aligned}\dot{\nabla}\mathbf{T} &= \frac{\partial T^i}{\partial t}\mathbf{g}_i + V^j\Gamma^k_{ij}T^i\mathbf{g}_k - V^\alpha(\nabla_\alpha T^i)\mathbf{g}_i\\ &= \left(\frac{\partial T^k}{\partial t} + V^j\Gamma^k_{ij}T^i - V^\alpha\nabla_\alpha T^k\right)\mathbf{g}_k\\ &\equiv \dot{\nabla}T^k\mathbf{g}_k\end{aligned} \tag{3.170}$$

The invariant time derivative of a contravariant tensor T^i is given from Eq. (3.170).

$$\dot{\nabla}T^i = \frac{\partial T^i}{\partial t} + V^j\Gamma^i_{jk}T^k - V^\alpha\nabla_\alpha T^i \tag{3.171}$$

Similarly, one obtains the invariant time derivative of a covariant tensor T_i

$$\dot{\nabla}T_i = \frac{\partial T_i}{\partial t} - V^j\Gamma^k_{ji}T_k - V^\alpha\nabla_\alpha T_i \tag{3.172}$$

The invariant time derivative of a mixed tensor T^i_j can be derived in

$$\begin{aligned}\dot{\nabla}T^i_j &= \frac{\partial T^i_j}{\partial t} + V^k\Gamma^i_{km}T^m_j - V^k\Gamma^n_{kj}T^i_n - V^\alpha\nabla_\alpha T^i_j\\ &= \frac{\partial T^i_j}{\partial t} + V^k(\Gamma^i_{km}T^m_j - \Gamma^n_{kj}T^i_n) - V^\alpha\nabla_\alpha T^i_j.\end{aligned} \tag{3.173}$$

The general invariant time derivative of a mixed fourth-order tensor can be derived in Grinfeld (2013).

$$\dot{\nabla} T^{i\alpha}_{j\beta} = \frac{\partial T^{i\alpha}_{j\beta}}{\partial t} - V^{\gamma} \nabla_{\gamma} T^{i\alpha}_{j\beta} + V^{p} \Gamma^{i}_{pq} T^{q\alpha}_{j\beta} - V^{p} \Gamma^{q}_{pj} T^{i\alpha}_{q\beta} \\ + \dot{\Gamma}^{\alpha}_{\delta} T^{i\delta}_{j\beta} - \dot{\Gamma}^{\delta}_{\beta} T^{i\alpha}_{j\delta} \tag{3.174}$$

where the time derivative of the Christoffel symbols for a moving surface is defined as

$$\dot{\Gamma}^{\alpha}_{\beta} = \nabla_{\beta} V^{\alpha} - P R^{\alpha}_{\beta} \tag{3.175}$$

in which P is the normal velocity in Eq. (3.165e) and R^{α}_{β} is the mean curvature of the moving surface.

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Chapter 4
Applications of Tensors and Differential Geometry

4.1 Nabla Operator in Curvilinear Coordinates

Nabla operator is a linear map of an arbitrary tensor into an image tensor in N-dimensional curvilinear coordinates. The Nabla operator can be usually defined in N-dimensional Cartesian coordinates $\{x^i\}$ using Einstein summation convention as

$$\nabla \equiv \mathbf{e}^i \frac{\partial}{\partial x^i} \quad \text{for} \quad i = 1, 2, \ldots, N \tag{4.1}$$

According to Eq. (2.12), the relation between the bases of Cartesian and general curvilinear coordinates can be written as

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial \mathbf{r}}{\partial x^j}\frac{\partial x^j}{\partial u^i} = \mathbf{e}_j \frac{\partial x^j}{\partial u^i} \tag{4.2}$$

Multiplying Eq. (4.2) by $\mathbf{g}^i \mathbf{e}^j$, one obtains the basis of Cartesian coordinates expressed in the curvilinear coordinate basis.

$$\mathbf{e}^j = \mathbf{g}^i \frac{\partial x^j}{\partial u^i} \tag{4.3}$$

Using the chain rule in the coordinate transformation, the Nabla operator in the general curvilinear coordinates $\{u^i\}$ results from Eq. (4.3) (Klingbeil 1966; Simmonds 1982).

$$\begin{aligned}\nabla \equiv \mathbf{e}^i \left(\frac{\partial}{\partial u^j}\frac{\partial u^j}{\partial x^i} \right) &= \mathbf{g}^k \frac{\partial x^i}{\partial u^k} \left(\frac{\partial}{\partial u^j}\frac{\partial u^j}{\partial x^i} \right) \\ &= \mathbf{g}^k \frac{\partial}{\partial u^j} \left(\frac{\partial x^i}{\partial u^k}\frac{\partial u^j}{\partial x^i} \right) = \mathbf{g}^k \frac{\partial}{\partial u^j} \left(\frac{\partial u^j}{\partial u^k} \right) \\ &= \mathbf{g}^k \frac{\partial}{\partial u^j} \left(\delta^j_k \right) = \mathbf{g}^k \frac{\partial}{\partial u^k}\end{aligned} \tag{4.4}$$

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Thus, the Nabla operator can be written in the curvilinear coordinates $\{u^i\}$ using Einstein summation convention.

$$\nabla \equiv \mathbf{g}^i \frac{\partial}{\partial u^i} = \mathbf{g}^i \nabla_i \quad \text{for} \quad i = 1, 2, \ldots, N \tag{4.5}$$

4.2 Gradient, Divergence, and Curl

Let ϕ be a velocity potential that exists only in a vortex-free flow; it can be defined as

$$\phi = \int v \, \mathrm{d}x \tag{4.6}$$

Differentiating Eq. (4.6) with respect to x, the velocity component results in

$$v = \frac{\partial \phi}{\partial x} \tag{4.7}$$

The velocity vector $\mathbf{v}$ can be written in the general curvilinear coordinates $\{u^i\}$ with the contravariant basis.

$$\mathbf{v} = v_i \mathbf{g}^i = \frac{\partial \phi}{\partial u^i} \mathbf{g}^i \tag{4.8}$$

4.2.1 Gradient of an Invariant

The gradient of an invariant ϕ (zero-order tensor) can be defined by

$$\begin{aligned} &\text{Grad } \phi = \nabla \phi = \mathbf{g}^i \frac{\partial \phi}{\partial u^i} \equiv \phi_{,i} \mathbf{g}^i = v_i \mathbf{g}^i = \mathbf{v} \\ &\Rightarrow \nabla \phi = \mathbf{v} = \left(\frac{\partial \phi}{\partial u^1} \mathbf{g}^1 + \frac{\partial \phi}{\partial u^2} \mathbf{g}^2 + \frac{\partial \phi}{\partial u^3} \mathbf{g}^3 \right) \end{aligned} \tag{4.9}$$

It is straightforward that the gradient of an invariant is a vector (first-order tensor).

4.2.2 Gradient of a Vector

The gradient of a contravariant vector **v** can be calculated using the covariant derivative of the covariant basis $\mathbf{g}_j$, as given in Eq. (2.158).

$$\begin{aligned}
\text{Grad } \mathbf{v} = \nabla \mathbf{v} &= \left(\mathbf{g}^i \frac{\partial}{\partial u^i} \right) (v^j \mathbf{g}_j) = \mathbf{g}^i \frac{\partial (v^j \mathbf{g}_j)}{\partial u^i} \\
&= \mathbf{g}^i \left(v^j_{,i} \mathbf{g}_j + v^j \mathbf{g}_{j,i} \right) = \left(v^k_{,i} \mathbf{g}_k + v^j \Gamma^k_{ij} \mathbf{g}_k \right) \mathbf{g}^i \\
&= \left(v^k_{,i} + v^j \Gamma^k_{ij} \right) \mathbf{g}_k \mathbf{g}^i \\
&\equiv v^k \big|_i \mathbf{g}_k \mathbf{g}^i
\end{aligned} \tag{4.10}$$

Analogously, the gradient of a covariant vector **v** can be written using the covariant derivative of the contravariant basis $\mathbf{g}^j$ in Eq. (2.189).

$$\begin{aligned}
\text{Grad } \mathbf{v} = \nabla \mathbf{v} &= \mathbf{g}^i \frac{\partial (v_j \mathbf{g}^j)}{\partial u^i} \\
&= \mathbf{g}^i \left(v_{j,i} \mathbf{g}^j + v_j \mathbf{g}^j_{,i} \right) = \left(v_{k,i} \mathbf{g}^k - v_j \Gamma^j_{ik} \mathbf{g}^k \right) \mathbf{g}^i \\
&= \left(v_{k,i} - v_j \Gamma^j_{ik} \right) \mathbf{g}^k \mathbf{g}^i \\
&\equiv v_k \big|_i \mathbf{g}^k \mathbf{g}^i
\end{aligned} \tag{4.11}$$

4.2.3 Divergence of a Vector

Let **v** be a vector in the curvilinear coordinates $\{u^i\}$; it can be written in the covariant basis $\mathbf{g}_j$.

$$\mathbf{v} = v^j \mathbf{g}_j \tag{4.12}$$

The divergence of **v** can be defined by

$$\begin{aligned}
\text{Div } \mathbf{v} = \nabla \cdot \mathbf{v} &= \left(\mathbf{g}^i \frac{\partial}{\partial u^i} \right) \cdot \mathbf{v} = \mathbf{g}^i \cdot \frac{\partial (v^j \mathbf{g}_j)}{\partial u^i} \\
&= \mathbf{g}^i \cdot \left(v^j_{,i} \mathbf{g}_j + v^j \mathbf{g}_{j,i} \right) \equiv \mathbf{g}^i \cdot \nabla_i \mathbf{v}
\end{aligned} \tag{4.13}$$

Using Eq. (2.158), the covariant derivative of the covariant basis $\mathbf{g}_j$ results in

$$\mathbf{g}_{j,i} = \Gamma^k_{ji} \mathbf{g}_k = \Gamma^k_{ij} \mathbf{g}_k \tag{4.14}$$

Substituting Eq. (4.14) into Eq. (4.13), one obtains the divergence of **v**.

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \mathbf{g}^i\cdot\left(v^j_{,i}\mathbf{g}_j + v^j\Gamma^k_{ij}\mathbf{g}_k\right)\\ &= \mathbf{g}^i\cdot\left(v^k_{,i}\mathbf{g}_k + v^j\Gamma^k_{ij}\mathbf{g}_k\right) = \mathbf{g}^i\cdot\left(v^k_{,i} + v^j\Gamma^k_{ij}\right)\mathbf{g}_k\\ &= \left(v^k_{,i} + v^j\Gamma^k_{ij}\right)\mathbf{g}^i\cdot\mathbf{g}_k \equiv v^k|_i\delta^i_k\\ &= \left(v^i_{,i} + v^j\Gamma^i_{ij}\right) = v^i|_i\end{aligned} \tag{4.15a}$$

According to Eq. (2.240), the second-kind Christoffel symbol can be rewritten as

$$\Gamma^i_{ij} = \frac{1}{J}\frac{\partial J}{\partial u^j} = \frac{\partial(\ln J)}{\partial u^j} \tag{4.15b}$$

Substituting Eq. (4.15b) into Eq. (4.15a), the divergence of $\mathbf{v}$ can be expressed in the Jacobian J.

$$\nabla\cdot\mathbf{v} = \frac{1}{J}\left(J\frac{\partial v^i}{\partial u^i} + v^i\frac{\partial J}{\partial u^i}\right) = \frac{1}{J}\frac{\partial(Jv^i)}{\partial u^i} = \frac{1}{J}\left(Jv^i\right)_{,i} \tag{4.15c}$$

Analogously, the covariant vector $\mathbf{v}$ can be written in the contravariant basis $\mathbf{g}^j$.

$$\mathbf{v} = v_j\mathbf{g}^j \tag{4.16}$$

The divergence of $\mathbf{v}$ can be derived using the contraction law and covariant derivative of the basis $\mathbf{g}^j$ in Eq. (2.189).

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \mathbf{g}^i\cdot\frac{\partial(v_j\mathbf{g}^j)}{\partial u^i} = \mathbf{g}^i\cdot\left(v_{j,i}\mathbf{g}^j + v_j\mathbf{g}^j_{,i}\right)\\ &= \mathbf{g}^i\cdot\left(v_{k,i}\mathbf{g}^k - v_j\Gamma^j_{ik}\mathbf{g}^k\right)\\ &= \left(v_{k,i} - v_j\Gamma^j_{ik}\right)\mathbf{g}^i\cdot\mathbf{g}^k\\ &\equiv v_k|_i\mathbf{g}^i\cdot\mathbf{g}^k = v_k|_i g^{ik} = v^m|_i g_{mk}g^{ik}\\ &= v^m|_i\delta^i_m = v^i|_i\end{aligned} \tag{4.17}$$

Some useful abbreviations are listed as follows:

- Divergence of the contravariant vector $\mathbf{v}$:

$$\begin{aligned}\nabla\cdot\mathbf{v} &\equiv \mathbf{g}^i\cdot\nabla_i\mathbf{v} = \mathbf{g}^i\cdot\nabla_i(v^j\mathbf{g}_j)\\ &= v^i_{,i} + v^j\Gamma^i_{ij} \equiv v^i|_i\\ &= \frac{1}{J}\frac{\partial(Jv^i)}{\partial u^i} = \frac{1}{J}\left(Jv^i\right)_{,i}\end{aligned} \tag{4.18a}$$

- Divergence of the covariant vector **v**:

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \mathbf{g}^i\cdot\nabla_i\mathbf{v} = \mathbf{g}^i\cdot\nabla_i(v_j\mathbf{g}^j)\\ &= \left(v_{k,i} - v_j\Gamma^j_{ik}\right)g^{ki} = v_k|_i g^{ki}\end{aligned} \tag{4.18b}$$

- Covariant derivative of the contravariant vector component:

$$v^k|_i \equiv v^k_{,i} + v^j\Gamma^k_{ij} = \frac{\partial v^k}{\partial u^i} + v^j\Gamma^k_{ij} \tag{4.19a}$$

- Covariant derivative of the covariant vector component:

$$v_k|_i \equiv \left(v_{k,i} - v_j\Gamma^j_{ki}\right) = \frac{\partial v_k}{\partial u^i} - v_j\Gamma^j_{ki} \tag{4.19b}$$

- Covariant derivative of the contravariant vector **v** with respect to u^i:

$$\nabla_i\mathbf{v} = \left(v^k_{,i} + v^j\Gamma^k_{ij}\right)\mathbf{g}_k = v^k|_i\mathbf{g}_k \tag{4.20a}$$

- Covariant derivative of the covariant vector **v** with respect to u^i:

$$\nabla_i\mathbf{v} = \left(v_{k,i} - v_j\Gamma^j_{ik}\right)\mathbf{g}^k = v_k|_i\mathbf{g}^k = v^m|_i g_{mk}\mathbf{g}^k. \tag{4.20b}$$

4.2.4 Divergence of a Second-Order Tensor

Let **T** be a contravariant tensor in the curvilinear coordinates $\{u^i\}$; it can be written in the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$.

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j \tag{4.21}$$

The divergence of **T** can be calculated from

$$\begin{aligned}\nabla\cdot\mathbf{T} &= \mathbf{g}^k\frac{\partial}{\partial u^k}\cdot\mathbf{T} = \mathbf{g}^k\cdot\frac{\partial(T^{ij}\mathbf{g}_i\mathbf{g}_j)}{\partial u^k}\\ &= \mathbf{g}^k\cdot\left(T^{ij}_{,k}\mathbf{g}_i\mathbf{g}_j + T^{ij}\mathbf{g}_{i,k}\mathbf{g}_j + T^{ij}\mathbf{g}_i\mathbf{g}_{j,k}\right)\end{aligned} \tag{4.22}$$

Using Eq. (2.158), the covariant derivative of the covariant basis $\mathbf{g}_i$ results in

$$\begin{aligned}\mathbf{g}_{i,k} &= \Gamma^m_{ik}\mathbf{g}_m = \Gamma^m_{ki}\mathbf{g}_m;\\ \mathbf{g}_{j,k} &= \Gamma^n_{jk}\mathbf{g}_n = \Gamma^n_{kj}\mathbf{g}_n\end{aligned}$$

Interchanging the indices, the divergence of a contravariant second-order tensor **T** becomes

$$\begin{aligned}\nabla \cdot \mathbf{T} &= \left(T^{ij}_{,k}\mathbf{g}_i\mathbf{g}_j + \Gamma^i_{km}T^{mj}\mathbf{g}_i\mathbf{g}_j + \Gamma^j_{km}T^{im}\mathbf{g}_i\mathbf{g}_j\right)\cdot \mathbf{g}^k \\ &= \left(T^{ij}_{,k}\delta^k_i + \Gamma^i_{km}T^{mj}\delta^k_i + \Gamma^j_{km}T^{im}\delta^k_i\right)\mathbf{g}_j\end{aligned} \tag{4.23}$$

Equation (4.23) can be written in the covariant basis $\mathbf{g}_j$ at $k = i$.

$$\begin{aligned}\nabla \cdot \mathbf{T} &= \left(T^{ij}_{,k} + \Gamma^i_{km}T^{mj} + \Gamma^j_{km}T^{im}\right)\delta^k_i\mathbf{g}_j \\ &= \left(T^{ij}_{,i} + \Gamma^i_{im}T^{mj} + \Gamma^j_{im}T^{im}\right)\mathbf{g}_j \\ &\equiv T^{ij}|_i\mathbf{g}_j\end{aligned} \tag{4.24a}$$

Using Eq. (2.240), the covariant derivative of the tensor component T^{ij} with respect u^i on the right-hand side (RHS) of Eq. (4.24a) can be expressed in the Jacobian J.

$$\begin{aligned}T^{ij}|_i &= T^{ij}_{,i} + \Gamma^i_{im}T^{mj} + \Gamma^j_{im}T^{im} \\ &= \frac{\partial T^{ij}}{\partial u^i} + T^{mj}\left(\frac{1}{J}\frac{\partial J}{\partial u^m}\right) + T^{im}\Gamma^j_{im} \\ &= T^{im}\Gamma^j_{im} + \frac{1}{J}\left(J\frac{\partial T^{ij}}{\partial u^i} + T^{ij}\frac{\partial J}{\partial u^i}\right) \\ &= T^{ik}\Gamma^j_{ik} + \frac{1}{J}\frac{\partial (JT^{ij})}{\partial u^i} = T^{ik}\Gamma^j_{ik} + J^{-1}\left(JT^{ij}\right)_{,i}\end{aligned} \tag{4.24b}$$

Therefore,

$$\nabla \cdot \mathbf{T} = T^{ij}|_i\mathbf{g}_j = \left(T^{ik}\Gamma^j_{ik} + J^{-1}\left(JT^{ij}\right)_{,i}\right)\mathbf{g}_j \tag{4.24c}$$

Interchanging the indices, the divergence of a covariant second-order tensor **T** can be written as

$$\begin{aligned}\nabla \cdot \mathbf{T} &= \mathbf{g}^k \cdot \frac{\partial (T_{ij}\mathbf{g}^i\mathbf{g}^j)}{\partial u^k} \\ &= \left(T_{ij,k}\mathbf{g}^i\mathbf{g}^j + T_{ij}\mathbf{g}^i_{,k}\mathbf{g}^j + T_{ij}\mathbf{g}^i\mathbf{g}^j_{,k}\right)\cdot \mathbf{g}^k \\ &= \left(T_{ij,k}\mathbf{g}^i\mathbf{g}^j - T_{ij}\Gamma^i_{km}\mathbf{g}^m\mathbf{g}^j - T_{ij}\Gamma^j_{km}\mathbf{g}^i\mathbf{g}^m\right)\cdot \mathbf{g}^k \\ &= \left(T_{ij,k} - T_{mj}\Gamma^m_{ki} - T_{im}\Gamma^m_{kj}\right)\mathbf{g}^i\left(\mathbf{g}^j \cdot \mathbf{g}^k\right) \\ &= T_{ij}|_k g^{jk}\mathbf{g}^i\end{aligned} \tag{4.25}$$

Furthermore, the divergence of a mixed second-order tensor **T** results as the same way at $k = i$.

$$\begin{aligned}\nabla \cdot \mathbf{T} &= \mathbf{g}^k \cdot \frac{\partial (T^i_j \mathbf{g}^j \mathbf{g}_i)}{\partial u^k} \\ &= \left(T^i_{j,k} + \Gamma^i_{km} T^m_j - \Gamma^m_{jk} T^i_m \right) \delta^k_i \mathbf{g}^j \\ &= \left(T^i_{j,i} + \Gamma^i_{im} T^m_j - \Gamma^m_{ji} T^i_m \right) \mathbf{g}^j \\ &\equiv T^i_j|_i \mathbf{g}^j \\ &= T^i_j|_i g^{kj} \mathbf{g}_k \end{aligned} \tag{4.26a}$$

Using Eq. (2.240), the covariant derivative of the mixed tensor component with respect to u^i on the RHS of Eq. (4.26a) can be written in the Jacobian J.

$$\begin{aligned} T^i_j|_i &= T^i_{j,i} + \Gamma^i_{im} T^m_j - \Gamma^m_{ij} T^i_m \\ &= \frac{1}{J} \left(J \frac{\partial T^i_j}{\partial u^i} + T^i_j \frac{\partial J}{\partial u^i} \right) - \Gamma^m_{ij} T^i_m \\ &= J^{-1} \left(J T^i_j \right)_{,i} - T^i_k \Gamma^k_{ij} \end{aligned} \tag{4.26b}$$

Therefore,

$$\begin{aligned} \nabla \cdot \mathbf{T} &= T^i_j|_i \mathbf{g}^j = T^i_j|_i g^{kj} \mathbf{g}_k \\ &= \left(J^{-1} (J T^i_j)_{,i} - T^i_k \Gamma^k_{ij} \right) g^{kj} \mathbf{g}_k \end{aligned} \tag{4.26c}$$

These results prove that the divergence of a second-order tensor **T**, such as the stress tensor **Π** or deformation tensor **D**, results in a first-order tensor, which is a vector in the curvilinear coordinates $\{u^i\}$.

4.2.5 Curl of a Covariant Vector

Let **v** be a covariant vector in the curvilinear coordinates $\{u^i\}$; it can be written in the contravariant basis $\mathbf{g}^j$.

$$\mathbf{v} = v_j \mathbf{g}^j \tag{4.27}$$

The curl (rotation) of **v** can be defined by

$$\begin{aligned}\text{Curl } \mathbf{v} &\equiv \text{Rot } \mathbf{v} \equiv \nabla \times \mathbf{v} \\ &= \mathbf{g}^i \frac{\partial}{\partial u^i} \times (v_j \mathbf{g}^j) = \mathbf{g}^i \times \frac{\partial (v_j \mathbf{g}^j)}{\partial u^i} \\ &= \mathbf{g}^i \times (v_{j,i}\mathbf{g}^j + v_j \mathbf{g}^j_{,i}) \\ &= v_{j,i}(\mathbf{g}^i \times \mathbf{g}^j) + v_j(\mathbf{g}^i \times \mathbf{g}^j_{,i})\end{aligned} \tag{4.28}$$

Using Eq. (2.189), the covariant derivative of the contravariant basis $\mathbf{g}^j$ results in

$$\mathbf{g}^j_{,i} = -\Gamma^j_{ik}\mathbf{g}^k \tag{4.29}$$

Substituting Eq. (4.29) into Eq. (4.28), one obtains the curl of $\mathbf{v}$.

$$\begin{aligned}\nabla \times \mathbf{v} &= v_{j,i}(\mathbf{g}^i \times \mathbf{g}^j) + v_j(\mathbf{g}^i \times \mathbf{g}^j_{,i}) \\ &= \hat{\varepsilon}^{ijk} v_{j,i}\mathbf{g}_k - v_j \Gamma^j_{ik}(\mathbf{g}^i \times \mathbf{g}^k) \\ &= \hat{\varepsilon}^{ijk} v_{j,i}\mathbf{g}_k - \hat{\varepsilon}^{ikm} v_j \Gamma^j_{ik}\mathbf{g}_m\end{aligned} \tag{4.30}$$

where the contravariant permutation symbols can be defined as (cf. Appendix A).

$$\hat{\varepsilon}^{ijk} = \begin{cases} +J^{-1} & \text{if} \quad (i,j,k) \text{ is an even permutation} \\ -J^{-1} & \text{if} \quad (i,j,k) \text{ is an odd permutation} \\ 0 & \text{if} \quad i=j, \quad \text{or} \quad i=k; \quad \text{or} \quad j=k \end{cases} \tag{4.31}$$

However, the second term in RHS of Eq. (4.30) vanishes due to the symmetric Christoffel symbols with respect to the indices of i and k, and the anticyclic permutation property with respect to i, k, and m.

$$\begin{aligned}\hat{\varepsilon}^{ikm} v_j \Gamma^j_{ik}\mathbf{g}_m &= \frac{v_j}{J}\left(\Gamma^j_{ik} - \Gamma^j_{ki}\right)\mathbf{g}_m \\ &= \frac{v_j}{J}\left(\Gamma^j_{ik} - \Gamma^j_{ik}\right)\mathbf{g}_m = 0\end{aligned} \tag{4.32}$$

Therefore, the curl of $\mathbf{v}$ in Eq. (4.30) becomes

$$\nabla \times \mathbf{v} = \hat{\varepsilon}^{ijk} v_{j,i}\mathbf{g}_k = \hat{\varepsilon}^{ijk} \frac{\partial v_j}{\partial u^i}\mathbf{g}_k. \tag{4.33}$$

4.3 Laplacian Operator

Laplacian operator is a linear map of an arbitrary tensor into an image tensor in N-dimensional curvilinear coordinates.

4.3.1 Laplacian of an Invariant

Laplacian of an invariant ϕ (zeroth-order tensor) is the divergence of (grad ϕ). Using Eq. (4.9), this expression can be written as

$$\text{Div (Grad } \phi) \equiv \nabla \cdot \nabla \phi = \nabla^2 \phi \equiv \Delta \phi \tag{4.34}$$

Substituting the gradient $\nabla \phi$ of Eq. (4.9) into Eq. (4.34), one obtains the Laplacian $\Delta \phi$.

$$\begin{aligned} \Delta \phi &\equiv \nabla \cdot \nabla \phi = \nabla \cdot (\phi_{,k} \mathbf{g}^k) = \nabla \cdot (v_k \mathbf{g}^k) \\ &= \mathbf{g}^l \cdot \frac{\partial}{\partial u^l} (\phi_{,k} \mathbf{g}^k) = \mathbf{g}^l \cdot (\phi_{,kl} \mathbf{g}^k + \phi_{,k} \mathbf{g}^k_{,l}) \end{aligned} \tag{4.35}$$

Using Eq. (2.189), the covariant derivative of the contravariant basis $\mathbf{g}^j$ results in

$$\mathbf{g}^k_{,l} = -\Gamma^k_{lm} \mathbf{g}^m \tag{4.36}$$

Inserting Eq. (4.36) into Eq. (4.35) and using Eq. (4.19b), the Laplacian of ϕ can be computed as

$$\begin{aligned} \Delta \phi &= \nabla^2 \phi \\ &= \mathbf{g}^l \cdot (\phi_{,kl} \mathbf{g}^k + \phi_{,k} \mathbf{g}^k_{,l}) = (\phi_{,kl} \mathbf{g}^k - \phi_{,k} \Gamma^k_{lm} \mathbf{g}^m) \cdot \mathbf{g}^l \\ &= (\phi_{,kl} \mathbf{g}^k - \phi_{,m} \Gamma^m_{lk} \mathbf{g}^k) \cdot \mathbf{g}^l \\ &= (\phi_{,kl} - \phi_{,m} \Gamma^m_{lk}) \mathbf{g}^k \cdot \mathbf{g}^l \\ &= (\phi_{,kl} - \phi_{,m} \Gamma^m_{kl}) g^{kl} \end{aligned} \tag{4.37}$$

The covariant vector components and their covariant derivatives with respect to u^k and u^l are defined as

$$\begin{aligned} &\phi_{,k} = \frac{\partial \phi}{\partial u^k} = v_k; \quad \phi_{,m} = \frac{\partial \phi}{\partial u^m} = v_m; \quad \phi_{,kl} = \frac{\partial^2 \phi}{\partial u^k \partial u^l} = v_{k,l} \\ &\Rightarrow \quad \Delta \phi = (v_{k,l} - v_m \Gamma^m_{kl}) g^{kl} \equiv v_k|_l g^{kl}. \end{aligned} \tag{4.38}$$

4.3.2 Laplacian of a Contravariant Vector

Laplacian of a contravariant vector (first-order tensor) is the divergence of grad $\mathbf{v}$ that can be computed as (Iben 1999)

$$
\begin{aligned}
\mathrm{Div}(\mathrm{Grad}\,\mathbf{v}) &= \Delta\mathbf{v} \equiv \nabla\cdot\nabla\mathbf{v} = \nabla^2\mathbf{v} \\
&= \left(v^k\big|_{l,m} - v^k\big|_p \Gamma^p_{lm} + v^p|_l \Gamma^k_{pm} \right) g^{lm}\mathbf{g}_k \\
&\equiv v^k\big|_{lm} g^{lm}\mathbf{g}_k
\end{aligned}
\tag{4.39}
$$

Obviously, the Laplacian of a contravariant vector results in another contravariant vector according to Eq. (4.39).

The second covariant derivative of the contravariant vector component v^k in Eq. (4.39) can be defined as

$$
v^k|_{lm} \equiv v^k\big|_{l,m} - v^k\big|_p \Gamma^p_{lm} + v^p|_l \Gamma^k_{pm} \tag{4.40}
$$

where

$$
v^k\big|_{l,m} = \left(v^k|_l\right)_{,m} \equiv v^k_{,lm} + v^n_{,m}\Gamma^k_{nl} + v^n\Gamma^k_{nl,m} \tag{4.41}
$$

$$
v^k\big|_p \equiv v^k_{,p} + v^n\Gamma^k_{np} \tag{4.42}
$$

$$
v^p|_l \equiv v^p_{,l} + v^n\Gamma^p_{nl} \tag{4.43}
$$

The vector triple product gives the relation of

$$
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{4.44}
$$

Thus, Eq. (4.44) can be rewritten in the curl identity of the vector $\mathbf{v}$ is set into the position of the vector $\mathbf{c}$.

$$
\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla\cdot(\nabla\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \Delta\mathbf{v} \tag{4.45}
$$

Therefore, the Laplacian of a vector $\mathbf{v}$ results in

$$
\begin{aligned}
\Delta\mathbf{v} \equiv \nabla\cdot(\nabla\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla\times(\nabla\times\mathbf{v}) \Leftrightarrow \\
\mathrm{Div}(\mathrm{Grad}\,\mathbf{v}) = \mathrm{Grad}(\mathrm{Div}\,\mathbf{v}) - \mathrm{Curl}(\mathrm{Curl}\,\mathbf{v})
\end{aligned}
\tag{4.46}
$$

4.4 Applying Nabla Operators in Spherical Coordinates

Spherical coordinates (ρ, ϕ, θ) are orthogonal curvilinear coordinates in which the bases are mutually perpendicular but not unitary. Figure 4.1 shows a point P in the spherical coordinates (ρ, ϕ, θ) embedded in orthonormal Cartesian coordinates (x^1, x^2, x^3). However, the vector component changes as the spherical coordinates vary.

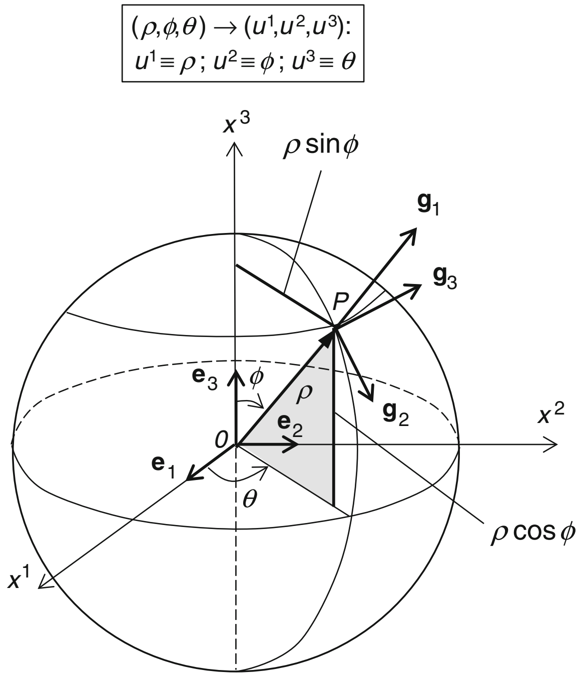

Fig. 4.1 Orthogonal spherical coordinates

The vector **OP** can be written in Cartesian coordinates (x^1, x^2, x^3):

$$\begin{aligned}\mathbf{R} &= (\rho \sin\phi \cos\theta)\mathbf{e}_1 + (\rho \sin\phi \sin\theta)\mathbf{e}_2 + \rho \cos\phi \; \mathbf{e}_3 \\ &\equiv x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3\end{aligned} \tag{4.47}$$

where

$\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$	are the orthonormal bases of Cartesian coordinates;
ϕ	is the equatorial angle;
θ	is the polar angle.

To simplify the formulation with Einstein symbol, the coordinates of u^1, u^2, and u^3 can be used for ρ, ϕ, and θ, respectively. Therefore, the coordinates of the point $P(u^1, u^2, u^3)$ can be written in Cartesian coordinates:

$$P(u^1, u^2, u^3) = \left\{ \begin{array}{l} x^1 = \rho \sin\phi \cos\theta \equiv u^1 \sin u^2 \cos u^3 \\ x^2 = \rho \sin\phi \sin\theta \equiv u^1 \sin u^2 \cos u^3 \\ x^3 = \rho \cos\phi \equiv u^1 \cos u^2 \end{array} \right\} \tag{4.48}$$

The covariant bases result from Chap. 2.

$$
\begin{aligned}
\mathbf{g}_1 &= (\sin\phi\cos\theta)\mathbf{e}_1 + (\sin\phi\sin\theta)\mathbf{e}_2 + \cos\phi\mathbf{e}_3 \Rightarrow |\mathbf{g}_1| = |\mathbf{g}_\rho| = 1 \\
\mathbf{g}_2 &= (\rho\cos\phi\cos\theta)\mathbf{e}_1 + (\rho\cos\phi\sin\theta)\mathbf{e}_2 - (\rho\sin\phi)\mathbf{e}_3 \Rightarrow |\mathbf{g}_2| = |\mathbf{g}_\phi| = \rho \\
\mathbf{g}_3 &= (-\rho\sin\phi\sin\theta)\mathbf{e}_1 + (\rho\sin\phi\cos\theta)\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow |\mathbf{g}_3| = |\mathbf{g}_\theta| = \rho\sin\phi
\end{aligned} \tag{4.49a}
$$

The covariant metric tensor **M** in the spherical coordinates can be computed from Eq. (4.49a).

$$
\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & (\rho\sin\phi)^2 \end{bmatrix} \tag{4.49b}
$$

Similarly, the contravariant bases result from Chap. 2.

$$
\begin{aligned}
\mathbf{g}^1 &= (\sin\phi\cos\theta)\mathbf{e}_1 + (\sin\phi\sin\theta)\mathbf{e}_2 + \cos\phi\mathbf{e}_3 \Rightarrow |\mathbf{g}^1| = 1 \\
\mathbf{g}^2 &= \left(\frac{1}{\rho}\cos\phi\cos\theta\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\cos\phi\sin\theta\right)\mathbf{e}_2 - \left(\frac{1}{\rho}\sin\phi\right)\mathbf{e}_3 \Rightarrow |\mathbf{g}^2| = \frac{1}{\rho} \\
\mathbf{g}^3 &= \left(-\frac{1}{\rho}\frac{\sin\theta}{\sin\phi}\right)\mathbf{e}_1 + \left(\frac{1}{\rho}\frac{\cos\theta}{\sin\phi}\right)\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \Rightarrow |\mathbf{g}^3| = \frac{1}{\rho\sin\phi}
\end{aligned} \tag{4.50a}
$$

The contravariant metric coefficients in the contravariant metric tensor $\mathbf{M}^{-1}$ can be calculated from Eq. (4.50a).

$$
\mathbf{M}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^{-2} & 0 \\ 0 & 0 & (\rho\sin\phi)^{-2} \end{bmatrix}. \tag{4.50b}
$$

4.4.1 Gradient of an Invariant

The gradient of an invariant $A \in \mathbf{R}$ can be written according to Eq. (4.9) in

$$
\nabla A = \mathbf{g}^i \frac{\partial A}{\partial u^i} \equiv A_{,i}\mathbf{g}^i \tag{4.51}
$$

Dividing the covariant basis by its vector length, the normalized covariant basis (covariant unitary basis) results in

$$
\mathbf{g}_i^* = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} = \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}} = \frac{\mathbf{g}_i}{h_i} \tag{4.52}
$$

The covariant basis in Eq. (4.52) is given in

$$\mathbf{g}_i = h_i \mathbf{g}_i^* \tag{4.53}$$

where h_i are the vector lengths, as given in Eq. (4.49a).

$$\begin{aligned} h_1 &= \sqrt{g_{11}} = |\mathbf{g}_1| \equiv |\mathbf{g}_\rho| = 1 \\ h_2 &= \sqrt{g_{22}} = |\mathbf{g}_2| \equiv |\mathbf{g}_\varphi| = \rho \\ h_3 &= \sqrt{g_{33}} = |\mathbf{g}_3| \equiv |\mathbf{g}_\theta| = \rho \sin\phi \end{aligned} \tag{4.54}$$

The contravariant bases can be transformed into the covariant bases in the orthogonal spherical contravariant basis, as given in Eq. (4.50b).

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j \Rightarrow \begin{cases} \mathbf{g}^1 = g^{11}\mathbf{g}_1 = \mathbf{g}_\rho \\ \mathbf{g}^2 = g^{22}\mathbf{g}_2 = \dfrac{1}{\rho^2}\mathbf{g}_\phi \\ \mathbf{g}^3 = g^{33}\mathbf{g}_3 = \dfrac{1}{(\rho\sin\phi)^2}\mathbf{g}_\theta \end{cases} \tag{4.55a}$$

Substituting Eqs. (4.53, 4.54) into Eq. (4.55a), one obtains

$$\begin{cases} \mathbf{g}^1 = \mathbf{g}_\rho = (h_1\mathbf{g}_\rho^*) = \mathbf{g}_\rho^* \\ \mathbf{g}^2 = \dfrac{1}{\rho^2}\mathbf{g}_\phi = \dfrac{1}{\rho^2}(h_2\mathbf{g}_\phi^*) = \dfrac{1}{\rho}\mathbf{g}_\phi^* \\ \mathbf{g}^3 = \dfrac{1}{(\rho\sin\phi)^2}\mathbf{g}_\theta = \dfrac{1}{(\rho\sin\phi)^2}(h_3\mathbf{g}_\theta^*) = \dfrac{1}{\rho\sin\phi}\mathbf{g}_\theta^* \end{cases} \tag{4.55b}$$

Using Eqs. (4.51, 4.55b), the gradient of A can be expressed in the physical vector components in the covariant unitary basis.

$$\begin{aligned} \nabla A &= \frac{\partial A}{\partial u^i}\mathbf{g}^i = \left(\frac{A_{,i}}{h_i}\right)\mathbf{g}_i^* \\ &= \frac{\partial A}{\partial \rho}\mathbf{g}_\rho^* + \frac{1}{\rho}\frac{\partial A}{\partial \phi}\mathbf{g}_\phi^* + \frac{1}{\rho\sin\phi}\frac{\partial A}{\partial \theta}\mathbf{g}_\theta^* \end{aligned} \tag{4.56}$$

4.4.2 Divergence of a Vector

The divergence of $\mathbf{v}$ can be computed using the Christoffel symbols described in Eq. (4.15a).

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \mathbf{g}^i \frac{\partial \mathbf{v}}{\partial u^i} = \mathbf{g}^i \frac{\partial (v^j \mathbf{g}_j)}{\partial u^i} \\ &= \left(v^i_{,i} + v^j \Gamma^i_{ij}\right) \equiv v^i|_i \end{aligned} \tag{4.57}$$

At first, the covariant derivatives of the contravariant vector components in Eq. (4.57) have to be computed.

$$\begin{aligned} v^1|_1 &= v^1_{,1} + \Gamma^1_{11}v^1 + \Gamma^1_{12}v^2 + \Gamma^1_{13}v^3 \quad \text{for} \quad i=1;\ j=1,2,3 \\ v^2|_2 &= v^2_{,2} + \Gamma^2_{21}v^1 + \Gamma^2_{22}v^2 + \Gamma^2_{23}v^3 \quad \text{for} \quad i=2;\ j=1,2,3 \\ v^3|_3 &= v^3_{,3} + \Gamma^3_{31}v^1 + \Gamma^3_{32}v^2 + \Gamma^3_{33}v^3 \quad \text{for} \quad i=3;\ j=1,2,3 \end{aligned} \tag{4.58}$$

The second-kind Christoffel symbols in spherical coordinates can be calculated as (Klingbeil 1966).

$$\begin{aligned} &\Gamma^1_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & -\rho\sin^2\phi \end{pmatrix}; \quad \Gamma^2_{ij} = \begin{pmatrix} 0 & \frac{1}{\rho} & 0 \\ \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & -\sin\phi\cos\phi \end{pmatrix}; \\ &\Gamma^3_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & \cot\phi \\ \frac{1}{\rho} & \cot\phi & 0 \end{pmatrix} \end{aligned} \tag{4.59}$$

The physical vector components v^{*i} result from the contravariant vector components in the covariant unitary basis $\mathbf{g}_i^*$ according to Eq. (B.11) in Appendix B.

$$v^i = \frac{v^{*i}}{h_i} \Rightarrow \begin{cases} v^1 = \dfrac{1}{h_1}v^{*1} \equiv v_\rho \\ v^2 = \dfrac{1}{h_2}v^{*2} \equiv \dfrac{1}{\rho}v_\phi \\ v^3 = \dfrac{1}{h_3}v^{*3} \equiv \dfrac{1}{\rho\sin\phi}v_\theta \end{cases} \tag{4.60}$$

Using Eqs. (4.59, 4.60), the covariant derivatives of the contravariant vector components can be computed as

$$\begin{aligned} v^1|_1 &= v^1_{,1} = \frac{\partial v_\rho}{\partial\rho} \\ v^2|_2 &= v^2_{,2} + \Gamma^2_{21}v^1 = v^2_{,2} + \frac{1}{\rho}v^1 = \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{1}{\rho}v_\rho \\ v^3|_3 &= v^3_{,3} + \Gamma^3_{31}v^1 + \Gamma^3_{32}v^2 = \frac{1}{\rho\sin\phi}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{\rho}v_\rho + \cot\phi\frac{v_\phi}{\rho} \end{aligned} \tag{4.61}$$

Thus, the divergence of $\mathbf{v}$ results from Eq. (4.61) in

$$\begin{aligned} \nabla\cdot\mathbf{v} &= v^i|_i \equiv v^1|_1 + v^2|_2 + v^3|_3 \\ &= \frac{\partial v_\rho}{\partial\rho} + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{1}{\rho\sin\phi}\frac{\partial v_\theta}{\partial\theta} + \frac{2v_\rho}{\rho} + \cot\phi\frac{v_\phi}{\rho}. \end{aligned} \tag{4.62}$$

4.4.3 Curl of a Vector

The curl of a vector results from Eq. (4.33).

$$\begin{aligned}\nabla \times \mathbf{v} &= \hat{\varepsilon}^{ijk} v_{j,i} \mathbf{g}_k \\ &= \frac{1}{J}\left[(v_{3,2} - v_{2,3})\mathbf{g}_1 + (v_{1,3} - v_{3,1})\mathbf{g}_2 + (v_{2,1} - v_{1,2})\mathbf{g}_3\right]\end{aligned} \tag{4.63}$$

The Jacobian of the spherical coordinates were calculated in Eq. (2.37) as

$$J = \rho^2 \sin\phi \tag{4.64}$$

Using Eqs. (B.19, 4.49b), the covariant vector components can be computed in their physical vector components.

$$v_i = g_{ij}\left(\frac{v^{*j}}{h_j}\right) \Rightarrow \begin{cases} v_1 = g_{11}\left(\frac{v^{*1}}{h_1}\right) = v_\rho \\ v_2 = g_{22}\left(\frac{v^{*2}}{h_2}\right) = \rho v_\phi \\ v_3 = g_{33}\left(\frac{v^{*3}}{h_3}\right) = \rho \sin\phi v_\theta \end{cases} \tag{4.65}$$

According to Eq. (4.53), the covariant bases can be written in the covariant unitary basis.

$$\mathbf{g}_i = h_i \mathbf{g}_i^* \Rightarrow \begin{cases} \mathbf{g}_1 = h_1 \mathbf{g}_1^* = 1.\mathbf{g}_1^* \equiv \mathbf{g}_\rho^* \\ \mathbf{g}_2 = h_2 \mathbf{g}_2^* = \rho \mathbf{g}_2^* \equiv \rho \mathbf{g}_\phi^* \\ \mathbf{g}_3 = h_3 \mathbf{g}_3^* = (\rho \sin\phi)\mathbf{g}_3^* \equiv (\rho \sin\phi)\mathbf{g}_\theta^* \end{cases} \tag{4.66}$$

Substituting Eqs. (4.64–4.66) into Eq. (4.63), the curl of **v** can be expressed in the unitary covariant basis $\mathbf{g}_i^*$.

$$\begin{aligned}\nabla \times \mathbf{v} &= \frac{1}{J}(v_{3,2} - v_{2,3})\mathbf{g}_1 + \frac{1}{J}(v_{1,3} - v_{3,1})\mathbf{g}_2 + \frac{1}{J}(v_{2,1} - v_{1,2})\mathbf{g}_3 \\ &= \left(\frac{\partial(\rho \sin\phi \cdot v_\theta)}{\partial\phi} - \frac{\partial(\rho \cdot v_\phi)}{\partial\theta}\right)\frac{1}{\rho^2 \sin\phi}\mathbf{g}_1^* \\ &\quad + \left(\frac{\partial v_\rho}{\partial\theta} - \frac{\partial(\rho \sin\phi \cdot v_\theta)}{\partial\rho}\right)\frac{1}{\rho \sin\phi}\mathbf{g}_2^* \\ &\quad + \left(\frac{\partial(\rho \cdot v_\phi)}{\partial\rho} - \frac{\partial v_\rho}{\partial\phi}\right)\frac{1}{\rho}\mathbf{g}_3^*\end{aligned} \tag{4.67}$$

Computing the partial derivatives in Eq. (4.67), one obtains the curl of **v** in the unitary spherical coordinate bases.

$$\begin{aligned}\nabla \times \mathbf{v} = & \left(\frac{1}{\rho}\frac{\partial v_\theta}{\partial \phi} - \frac{1}{\rho \sin\phi}\frac{\partial v_\phi}{\partial \theta} + \cot\phi \frac{v_\theta}{\rho}\right)\mathbf{g}_\rho^* \\ & + \left(\frac{1}{\rho \sin\phi}\frac{\partial v_\rho}{\partial \theta} - \frac{\partial v_\theta}{\partial \rho} - \frac{v_\theta}{\rho}\right)\mathbf{g}_\phi^* \\ & + \left(\frac{\partial v_\phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial v_\rho}{\partial \phi} + \frac{v_\phi}{\rho}\right)\mathbf{g}_\theta^*.\end{aligned} \tag{4.68}$$

4.5 The Divergence Theorem

4.5.1 Gauss and Stokes Theorems

The divergence theorem, known as *Gauss theorem* deals with the relation between the flow of a vector or tensor field through the closed surface and the characteristics of the vector (tensor) in the volume closed by the surface. Gauss law states that the flux of a vector through any closed surface is proportional to the charge in the volume closed by the surface. This divergence theorem is a very useful tool that can be mostly applied to engineering and physics, such as fluid dynamics and electrodynamics, to derive the Navier–Stokes equations and Maxwell's equations, respectively.

The Gauss theorem can be generally written in a three-dimensional space.

$$\oint_S \mathbf{v} \cdot \mathbf{n} dS = \int_V \nabla \cdot \mathbf{v} dV \tag{4.69}$$

where $\mathbf{v}$ is the fluid vector through the surface S; $\mathbf{n}$ is the normal vector on the surface; and $\nabla \cdot \mathbf{v}$ is the divergence of the vector $\mathbf{v}$ (Fig. 4.2).

The outward fluid flux from the volume V causes the negative change rate of the volume mass with time.

$$\oint_S \rho \mathbf{v} \cdot \mathbf{n} dS = -\int_V \frac{\partial \rho}{\partial t} dV \tag{4.70}$$

Using Gauss divergence theorem, the balance of mass (also continuity equation) in the control volume V can be derived in

$$\oint_S \rho \mathbf{v} \cdot \mathbf{n} dS = -\int_V \frac{\partial \rho}{\partial t} dV = \int_V \nabla \cdot (\rho \mathbf{v}) dV \tag{4.71}$$

where ρ is the fluid density.

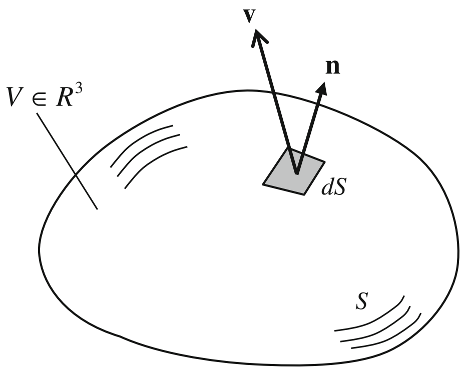

Fig. 4.2 Fluid flux through a closed surface S

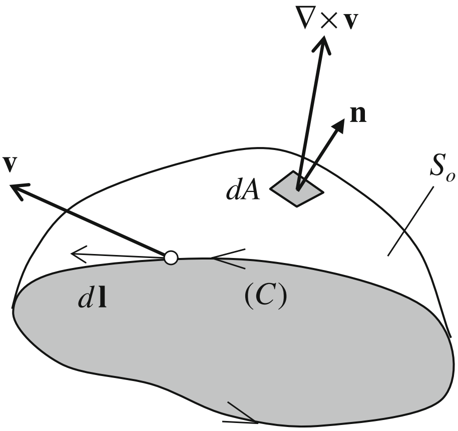

Fig. 4.3 Fluid flux through an open surface S_o

By rearranging the second and third terms in Eq. (4.71), the continuity equation can be written in the integral form for a control volume V:

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) \mathrm{d}V = 0 \Leftrightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{4.72}$$

Stokes theorem can be used for an open surface S_o, as shown in Fig. 4.3. The Stokes theorem indicates that the flow velocity along the closed curve (C) is equal to the flux of curl **v** going through the open surface S_o.

$$\oint_{(C)} \mathbf{v} \cdot \mathrm{d}\mathbf{l} = \int_{S_o} (\nabla \times \mathbf{v}) \cdot \mathbf{n} \mathrm{d}A \tag{4.73}$$

where **dl** is the length differential on the closed curve (C) of the surface S_o; $\nabla \times \mathbf{v}$ is the curl of the vector **v**.

4.5.2 Green's Identities

The Green's identities can be derived from the Gauss divergence theorem. Sometimes, they can be usefully applied to the boundary element method (BEM) using the Green's function (Nguyen-Schäfer 2013; Crocker 2007; Fahy and Gardonio 2007). Two Green's identities are discussed in the following section.

4.5.3 First Green's Identity

The vector $\mathbf{v}$ can be chosen as the product of two arbitrary scalars ψ and φ.

$$\mathbf{v} = \psi \nabla \varphi \tag{4.74}$$

The divergence of $\mathbf{v}$ can be computed as follows:

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \nabla \cdot (\psi \nabla \varphi) \\ &= \nabla \psi \cdot \nabla \varphi + \psi \nabla \nabla \varphi \\ &= \nabla \psi \cdot \nabla \varphi + \psi \nabla^2 \varphi \end{aligned} \tag{4.75}$$

Applying the Gauss divergence law of Eqs. (4.69–4.75), one obtains

$$\begin{aligned} &\int_V \nabla \cdot \mathbf{v} \mathrm{d}V = \oint_S \mathbf{v} \cdot \mathbf{n} \mathrm{d}S \Leftrightarrow \\ &\int_V \left(\nabla \psi \cdot \nabla \varphi + \psi \nabla^2 \varphi \right) \mathrm{d}V = \oint_S \psi (\nabla \varphi \cdot \mathbf{n}) \mathrm{d}S = \oint_S \psi \frac{\partial \varphi}{\partial n} \mathrm{d}S \end{aligned} \tag{4.76}$$

4.5.4 Second Green's Identity

The vector $\mathbf{v}$ can be chosen as the function of two arbitrary scalar products of ψ and φ.

$$\mathbf{v} = \psi \nabla \varphi - \varphi \nabla \psi \tag{4.77}$$

Thus, the divergence of $\mathbf{v}$ can be calculated as follows:

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \nabla \cdot (\psi \nabla \varphi - \varphi \nabla \psi) \\ &= \nabla \psi \cdot \nabla \varphi + \psi \nabla^2 \varphi - \nabla \varphi \cdot \nabla \psi - \varphi \nabla^2 \psi \\ &= \psi \nabla^2 \varphi - \varphi \nabla^2 \psi \end{aligned} \tag{4.78}$$

Applying the Gauss divergence law of Eq. (4.69–4.78), one obtains

$$\begin{aligned}&\int_V \nabla\cdot\mathbf{v}\mathrm{d}V=\oint_S \mathbf{v}\cdot\mathbf{n}\mathrm{d}S \Leftrightarrow\\&\int_V \left(\psi\nabla^2\varphi-\varphi\nabla^2\psi\right)\mathrm{d}V=\oint_S \left(\psi\nabla\varphi-\varphi\nabla\psi\right)\cdot\mathbf{n}dS\\&=\oint_S\left(\psi\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial\psi}{\partial n}\right)\mathrm{d}S\end{aligned} \tag{4.79}$$

where the gradients of the scalars in the normal direction can be defined as

$$\frac{\partial\varphi}{\partial n}\equiv\nabla\varphi\cdot\mathbf{n};\quad \frac{\partial\psi}{\partial n}\equiv\nabla\psi\cdot\mathbf{n}. \tag{4.80}$$

4.5.5 Differentials of Area and Volume

In Gauss divergence theorem, the differentials of area dA and volume dV are changed from Cartesian coordinates to other general curvilinear coordinates by coordinate transformations.

4.5.6 Calculating the Differential of Area

Figure 4.4 shows the transformation of Cartesian coordinates $\{x^i\}$ into the curvilinear coordinates $\{u^i\}$. The differential d**r** can be written in the covariant basis.

$$\mathrm{d}\mathbf{r}=\frac{\partial\mathbf{r}}{\partial u^i}\mathrm{d}u^i=\mathbf{g}_i\mathrm{d}u^i \tag{4.81}$$

Therefore,

$$\mathrm{d}\mathbf{x}^i=\mathbf{e}_i\mathrm{d}x^i=\mathbf{g}_i\mathrm{d}u^i\Leftrightarrow\begin{cases}\mathrm{d}\mathbf{x}^1=\mathbf{e}_1\mathrm{d}x^1=\mathbf{g}_1\mathrm{d}u^1\\ \mathrm{d}\mathbf{x}^2=\mathbf{e}_2\mathrm{d}x^2=\mathbf{g}_2\mathrm{d}u^2\end{cases} \tag{4.82}$$

The area differential can be calculated in the curvilinear coordinates (Nayak 2012).

$$\begin{aligned}\mathrm{d}A&=\left|\mathrm{d}\mathbf{x}^1\times d\mathbf{x}^2\right|=\left|(\mathbf{e}_1\times\mathbf{e}_2)\right|\mathrm{d}x^1\mathrm{d}x^2=\mathrm{d}x^1\mathrm{d}x^2\\&=\left|(\mathbf{g}_1\times\mathbf{g}_2)\right|\mathrm{d}u^1\mathrm{d}u^2\end{aligned} \tag{4.83}$$

Using the Lagrange identity in Appendix E, one has the relation of

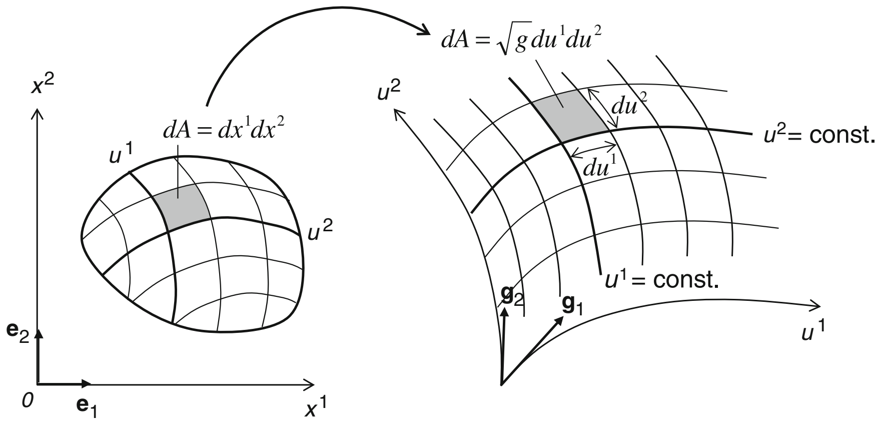

Fig. 4.4 Coordinate transformation in two-dimensional coordinates

$$\begin{aligned} |(\mathbf{g}_1 \times \mathbf{g}_2)| &= \sqrt{g_{11}g_{22} - (g_{12})^2} \\ &\equiv \sqrt{g} \end{aligned} \tag{4.84}$$

where g_{ij} are the covariant metric coefficients, as defined in Eq. (2.52).

Substituting Eq. (4.84) into Eq. (4.83), the area differential becomes

$$\begin{aligned} \mathrm{d}A &= \mathrm{d}x^1 \mathrm{d}x^2 \\ &= |(\mathbf{g}_1 \times \mathbf{g}_2)| \mathrm{d}u^1 \mathrm{d}u^2 \\ &= \sqrt{g_{11}g_{22} - (g_{12})^2} \mathrm{d}u^1 \mathrm{d}u^2 \\ &\equiv \sqrt{g} \mathrm{d}u^1 \mathrm{d}u^2. \end{aligned} \tag{4.85}$$

4.5.7 Calculating the Differential of Volume

The volume differential can be calculated in the curvilinear coordinates (Fig. 4.5).

$$\begin{aligned} \mathrm{d}V &= \left|(\mathrm{d}\mathbf{x}^1 \times \mathrm{d}\mathbf{x}^2) \cdot \mathrm{d}\mathbf{x}^3\right| = |(\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3| \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \\ &= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 = \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \\ &= |(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3| \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 = [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 \\ &= J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 \end{aligned} \tag{4.86}$$

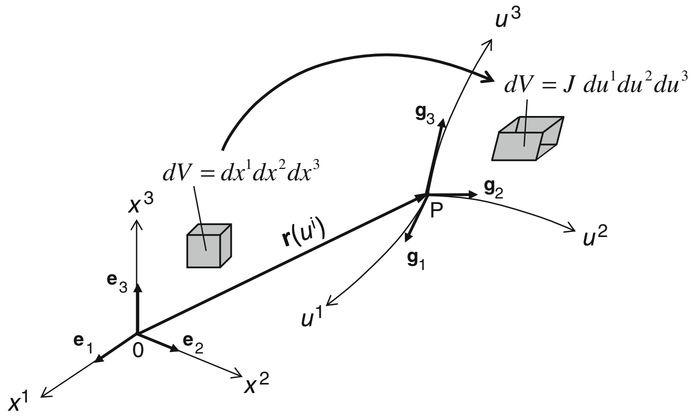

Fig. 4.5 Coordinate transformation in three-dimensional coordinates

The scalar triple product of the covariant bases of the curvilinear coordinates can be defined as

$$(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] \tag{4.87}$$

The determinant of the covariant basis tensor equals the scalar triple product of the covariant bases, as given in Eq. (1.10).

$$[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \frac{\partial x^1}{\partial u^3} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^2}{\partial u^3} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} = J \tag{4.88}$$

where J (Jacobian) is the determinant of the covariant basis tensor.

Using Eqs. (4.15a, 4.85, 4.86), the Gauss divergence theorem in general curvilinear coordinates $u^i \in \mathbf{R}^3$ can be written in the tensor integral equation:

$$\begin{aligned} \oint_S \mathbf{v} \cdot \mathbf{n} \mathrm{d}S = \int_V \nabla \cdot \mathbf{v} \mathrm{d}V \Leftrightarrow \oint_S v^i n_i \sqrt{g} \mathrm{d}u^1 \mathrm{d}u^2 \\ = \int_V (v^i_{,i} + v^j \Gamma^i_{ij}) J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 \equiv \int_V v^i|_i J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 \end{aligned} \tag{4.89}$$

The physical vector components v^{*i} result from the contravariant vector components in the normalized covariant basis (unitary basis) $\mathbf{g}_i^*$ according to Eq. (B.11) in Appendix B.

$$v^i = \frac{v^{*i}}{h_i} \quad \text{for} \quad i = 1, 2, 3 \tag{4.90}$$

The scale factor h_i can be defined as the covariant basis norm $|\mathbf{g}_i|$ of the curvilinear coordinates $\{u^i\}$.

$$h_i = |\mathbf{g}_i| = \sqrt{g_{(ii)}} \text{ (no summation over } i\text{)} \tag{4.91}$$

Therefore, the divergence theorem in Eq. (4.89) can be rewritten in the curvilinear coordinates u^i with the physical vector components as follows:

$$\begin{aligned} \oint_S \frac{v^{*i}}{h_i} n_i \sqrt{g} du^1 du^2 &= \int_V \left(\frac{v^{*i}_{,i}}{h_i} + \frac{v^{*j}}{h_j} \Gamma^i_{ij} \right) J du^1 du^2 du^3 \\ &\equiv \int_V \frac{1}{h_i} v^{*i}\big|_i J du^1 du^2 du^3 \end{aligned} \tag{4.92}$$

In the following sections, some applications of tensor analysis and differential geometry are applied to computational fluid dynamics (CFD), continuum mechanics, classical electrodynamics, electrodynamics in relativity fields, and the Einstein field theory as well.

4.6 Governing Equations of Computational Fluid Dynamics

Navier–Stokes equations describe fluid flows in Computational Fluid Dynamics (CFD). In this book, Navier–Stokes equations are derived for compressible flows in a general rotating frame of the turbomachinery. The rotating frame rotates at an angular velocity $\boldsymbol{\omega}(t)$ with respect to the inertial coordinate system (Chen 2010; Schobeiri 2012). At $\boldsymbol{\omega} = \mathbf{0}$, the Navier–Stokes equations can be used for a non-rotating frame (a special case). In this case, the velocity $\mathbf{w}$ in Eq. (4.94) is used in the equations instead of the absolute fluid velocity $\mathbf{v}$.

4.6.1 Continuity Equation

The continuity equation satisfies the mass balance of the fluid in the given control volume. According to Eq. (4.72), the continuity equation can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{4.93}$$

The absolute velocity $\mathbf{v}$ is the sum of the relative velocity $\mathbf{w}$ and the circumferential velocity $\mathbf{u}$.

$$\mathbf{v} = \mathbf{w} + \mathbf{u} = \mathbf{w} + (\boldsymbol{\omega} \times \mathbf{r}) \tag{4.94}$$

Substituting the absolute velocity $\mathbf{v}$ in Eq. (4.94) into Eq. (4.93), one obtains the continuity equation of compressible fluids in a rotating frame.

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot \rho(\mathbf{w} + (\boldsymbol{\omega} \times \mathbf{r})) = 0 \Leftrightarrow \\
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) + \nabla \cdot \rho(\boldsymbol{\omega} \times \mathbf{r}) = 0 \Leftrightarrow \\
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) + \rho \nabla \cdot (\boldsymbol{\omega} \times \mathbf{r}) + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \rho = 0
\end{aligned} \tag{4.95}$$

The third term in Eq. (4.95) equals zero because the circumferential velocity $(\boldsymbol{\omega} \times \mathbf{r})$ does not change in the inertial coordinates at an arbitrary radius vector $\mathbf{r}$.

$$\rho \nabla \cdot (\boldsymbol{\omega} \times \mathbf{r}) = 0 \tag{4.96}$$

Thus, Eq. (4.95) becomes

$$\begin{aligned}
&\left(\frac{\partial \rho}{\partial t} + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \rho \right) + \nabla \cdot (\rho \mathbf{w}) \\
&\equiv \frac{\partial_R \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) = 0
\end{aligned} \tag{4.97}$$

The partial derivative with respect to the rotating frame can be defined by Schobeiri (2012).

$$\frac{\partial_R \rho}{\partial t} \equiv \frac{\partial \rho}{\partial t} + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \rho \tag{4.98}$$

The continuity Eq. (4.97) can be written in the tensor equation of

$$\begin{aligned}
&\frac{\partial_R \rho}{\partial t} + \nabla \cdot (\rho \mathbf{w}) = 0 \\
&\Leftrightarrow \frac{\partial_R \rho}{\partial t} + (\rho w^i)|_i = 0 \\
&\Leftrightarrow \frac{\partial_R \rho}{\partial t} + (\rho w^i)_{,i} + (\rho w^j)\Gamma^i_{ij} = 0
\end{aligned} \tag{4.99a}$$

According to Eq. (4.18a), the continuity equation can be written in the Jacobian J.

$$\begin{aligned} &\frac{\partial_R \rho}{\partial t} + (\rho w^i)|_i = 0 \\ \Leftrightarrow &\frac{\partial_R \rho}{\partial t} + \frac{1}{J}\frac{\partial (J\rho w^i)}{\partial u^i} = 0 \\ \Leftrightarrow &\frac{\partial_R \rho}{\partial t} + \frac{1}{J}(J\rho w^i)_{,i} = 0 \end{aligned} \tag{4.99b}$$

The physical vector component in the normalized covariant basis $\mathbf{g}_i^*$ (unitary basis) can be obtained from Eq. (B.11) in Appendix B.

$$w^{*i} = h_i w^i \Rightarrow w^i = \frac{w^{*i}}{h_i} \tag{4.100}$$

where h_i is the norm of the covariant basis $\mathbf{g}_i$.

Therefore, the continuity tensor Eq. (4.99a) can be written in the physical velocity components of w^* using Eqs. (2.244, 4.100).

$$\begin{aligned} &\frac{\partial_R \rho}{\partial t} + \left(\rho \frac{w^{*i}}{h_i}\right)_{,i} + \left(\rho \frac{w^{*j}}{h_j}\right)\Gamma^i_{ij} = 0 \\ \Leftrightarrow &\frac{\partial_R \rho}{\partial t} + \left(\rho \frac{w^{*i}}{h_i}\right)_{,i} + \frac{1}{J}\left(\rho \frac{w^{*j}}{h_j}\right)\frac{\partial J}{\partial u^j} = 0 \end{aligned} \tag{4.101a}$$

and using Eq. (4.100), the continuity tensor Eq. (4.99b) becomes

$$\begin{aligned} &\frac{\partial_R \rho}{\partial t} + \frac{1}{J}\frac{\partial (J\rho w^i)}{\partial u^i} = 0 \Leftrightarrow \frac{\partial_R \rho}{\partial t} + \frac{1}{J}\left(J\rho \frac{w^{*i}}{h_i}\right)_{,i} = 0 \\ \Leftrightarrow &\frac{\partial_R \rho}{\partial t} + \left(\rho \frac{w^{*i}}{h_i}\right)_{,i} + \frac{1}{J}\left(\rho \frac{w^{*i}}{h_i}\right)\frac{\partial J}{\partial u^i} = 0. \end{aligned} \tag{4.101b}$$

4.6.2 Momentum Equations

The momentum equations describe the balance of forces in a given control volume V. According to Newton's second law, the balance of forces $\mathbf{F}_i$ for an arbitrary unit volume can be written as

$$\rho \frac{D\mathbf{v}}{Dt} = \sum_{i=1}^{n} \mathbf{F}_i \tag{4.102}$$

The substantial derivative in the inertial coordinate system is defined as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{4.103}$$

where the first term is the time derivative; $\mathbf{v}$ is the absolute velocity.

Substituting the absolute velocity $\mathbf{v}$ in Eq. (4.94) into Eq. (4.103), one obtains

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} &= \frac{D(\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r})}{Dt} = \frac{\partial(\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r})}{\partial t} + \mathbf{v} \cdot \nabla(\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \frac{\partial \mathbf{w}}{\partial t} + \frac{\partial(\boldsymbol{\omega} \times \mathbf{r})}{\partial t} + (\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla(\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r}) \end{aligned} \tag{4.104}$$

The second term in the RHS of Eq. (4.104) can be calculated as

$$\frac{\partial(\boldsymbol{\omega} \times \mathbf{r})}{\partial t} = \left(\boldsymbol{\omega} \times \frac{\partial \mathbf{r}}{\partial t}\right) + \left(\frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{r}\right) \tag{4.105}$$

The first term in the RHS of Eq. (4.105) equals zero because the radius vector $\mathbf{r}$ does not vary with time. Thus, Eq. (4.105) simply becomes

$$\frac{\partial(\boldsymbol{\omega} \times \mathbf{r})}{\partial t} = \left(\frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{r}\right) \tag{4.106}$$

The third term in the RHS of Eq. (4.104) can be written as

$$\begin{aligned} \mathbf{v} \cdot \nabla \mathbf{v} &= (\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla(\mathbf{w} + \boldsymbol{\omega} \times \mathbf{r}) \\ &= \mathbf{w} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla(\boldsymbol{\omega} \times \mathbf{r}) + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \mathbf{w} + (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla(\boldsymbol{\omega} \times \mathbf{r}) \end{aligned} \tag{4.107a}$$

where the last three terms in the RHS of Eq. (4.107a) result from (Chen 2010; Schobeiri 2012):

$$\begin{aligned} \mathbf{w} \cdot \nabla(\boldsymbol{\omega} \times \mathbf{r}) &= \boldsymbol{\omega} \times \mathbf{w}; \\ (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla \mathbf{w} &= \boldsymbol{\omega} \times \mathbf{w}; \\ (\boldsymbol{\omega} \times \mathbf{r}) \cdot \nabla(\boldsymbol{\omega} \times \mathbf{r}) &= \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \end{aligned} \tag{4.107b}$$

Substituting Eqs. (4.106, 4.107a, 4.107b) into Eq. (4.104), the substantial derivative of the absolute velocity $\mathbf{v}$ results in

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{w}}{\partial t} + \mathbf{w} \cdot \nabla \mathbf{w} + \left(\frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{r}\right) + [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] + 2(\boldsymbol{\omega} \times \mathbf{w}) \tag{4.108}$$

The substantial derivative of the absolute velocity **v** consists of the following terms: the first term in the RHS of Eq. (4.108) is the time derivative of the relative velocity **w**; the second term denotes the convection term; the third term is the circumferential acceleration at the radius **r**; the fourth term displays the centripetal acceleration of the fluid unit volume; and the last term is the Coriolis acceleration.

The external forces acting upon the fluid control volume comprise the pressure, fluid viscous, and gravity forces. As a result, the momentum equations for a rotating frame with an angular velocity $\boldsymbol{\omega}(t)$ result from Eqs. (4.102, 4.108).

$$\begin{aligned}&\frac{\partial \mathbf{w}}{\partial t}+\mathbf{w}\cdot\nabla\mathbf{w}+\left(\frac{\partial \boldsymbol{\omega}}{\partial t}\times\mathbf{r}\right)+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+2(\boldsymbol{\omega}\times\mathbf{w})\\ &\quad=-\frac{\nabla p}{\rho}+\frac{\nabla\cdot\boldsymbol{\Pi}}{\rho}-\nabla(gz)\end{aligned} \tag{4.109}$$

where $\boldsymbol{\Pi}$ is the viscous stress tensor of fluid that is written using the Stokes relation (Chen 2010).

$$\begin{aligned}\boldsymbol{\Pi}&=2\mu\mathbf{E}-\frac{2}{3}\mu\nabla\cdot(\mathbf{w}\mathbf{E})\\ &=\pi^{ij}\mathbf{g}_i\mathbf{g}_j=\pi_{ij}\mathbf{g}^i\mathbf{g}^j\end{aligned} \tag{4.110a}$$

in which μ is the dynamic viscosity of fluid; $\mathbf{E}$ is the strain tensor of fluid that is written in the strain components, cf. Eq. (4.176):

$$\mathbf{E}=\varepsilon^{ij}\mathbf{g}_i\mathbf{g}_j=\varepsilon_{ij}\mathbf{g}^i\mathbf{g}^j \tag{4.110b}$$

The first two terms in the left-hand side (LHS) of Eq. (4.109) can be written in the tensor equation with the physical components as

$$\begin{aligned}\frac{\partial \mathbf{w}}{\partial t}+\mathbf{w}\cdot\nabla\mathbf{w}&=\frac{\partial(w^k\mathbf{g}_k)}{\partial t}+(w^j\mathbf{g}_j)\cdot\nabla\mathbf{w}\\ &=\dot{w}^k\mathbf{g}_k+(w^j\mathbf{g}_j)\cdot\left(w^k\big|_i\mathbf{g}_k\mathbf{g}^i\right)=\dot{w}^k\mathbf{g}_k+w^jw^k\Big|_i\delta^i_j\mathbf{g}_k\\ &=\dot{w}^k\mathbf{g}_k+w^kw^i\big|_i\mathbf{g}_k=\left(\dot{w}^k+w^kw^i\big|_i\right)\mathbf{g}_k\end{aligned} \tag{4.111}$$

Equation (4.111) can be written in the unitary covariant basis with the physical velocity components of w^* using Eq. (4.100).

$$\begin{aligned}\frac{\partial \mathbf{w}}{\partial t}+\mathbf{w}\cdot\nabla\mathbf{w}&=h_{\mathrm{k}}\left(\frac{\dot{w}^{*\mathrm{k}}}{h_{\mathrm{k}}}+\frac{w^{*\mathrm{k}}}{h_{\mathrm{k}}h_i}w^{*i}\Big|_i\right)\mathbf{g}^*_{\mathrm{k}}\\ &=\left(\dot{w}^{*\mathrm{k}}+\frac{1}{h_{\mathrm{i}}}w^{*\mathrm{k}}w^{*\mathrm{i}}\Big|_{\mathrm{i}}\right)\mathbf{g}^*_{\mathrm{k}}\end{aligned} \tag{4.112}$$

where h_i and h_k are the norms of the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_k$, respectively.

The third term in the LHS of Eq. (4.109) can be written in the tensor equation of the unitary covariant basis with the physical vector components and the contravariant permutation symbols (cf. Appendix A).

$$\begin{aligned}\frac{\partial \boldsymbol{\omega}}{\partial t} \times \mathbf{r} &= \dot{\omega}_i r_j (\mathbf{g}^i \times \mathbf{g}^j) \\ &= \hat{\varepsilon}^{ijk} \dot{\omega}_i r_j \mathbf{g}_k \\ &= \hat{\varepsilon}^{ijk} (h_k \dot{\omega}_i r_j) \mathbf{g}_k^*\end{aligned} \tag{4.113}$$

The fourth term in the LHS of Eq. (4.109) can be written in the tensor equation and the covariant permutation symbols (cf. Appendix A).

$$\begin{aligned}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) &= (\omega_l \mathbf{g}^l) \times (\omega^i r^j \hat{\varepsilon}_{ijk} \mathbf{g}^k) \\ &= \omega_l \omega^i r^j \hat{\varepsilon}_{ijk} (\mathbf{g}^l \times \mathbf{g}^k) \\ &= \omega_l \omega^i r^j \hat{\varepsilon}_{ijk} \hat{\varepsilon}^{lkm} \mathbf{g}_m \\ &= \omega_l \omega^i r^j \delta_{ijk}^{lkm} \mathbf{g}_m \\ &= \omega_l \omega^i r^j \delta_{ij}^{lm} \mathbf{g}_m\end{aligned} \tag{4.114}$$

It can be written in the tensor equation of the unitary covariant basis with the physical vector components.

$$\begin{aligned}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) &= h_m \omega_l \omega^i r^j \delta_{ij}^{lm} \mathbf{g}_m^* \\ &= h_k \omega_l \omega^i r^j \delta_{ij}^{lk} \mathbf{g}_k^*\end{aligned} \tag{4.115}$$

The Coriolis term in the LHS of Eq. (4.109) can be written in the tensor equation of the unitary covariant basis with the physical vector components and the contravariant permutation symbols (cf. Appendix A).

$$\begin{aligned}2(\boldsymbol{\omega} \times \mathbf{w}) &= 2\omega_i w^m (\mathbf{g}^i \times \mathbf{g}_m) = 2\omega_i w^m (\mathbf{g}^i \times g_{mj} \mathbf{g}^j) \\ &= 2\hat{\varepsilon}^{ijk} \omega_i g_{mj} w^m \mathbf{g}_k \\ &= 2\hat{\varepsilon}^{ijk} \omega_i g_{mj} \frac{h_k}{h_m} w^{*m} \mathbf{g}_k^*\end{aligned} \tag{4.116}$$

Using Eqs. (B.2, B.3), the pressure tensor in the RHS of Eq. (4.109) can be written in the unitary covariant basis with the physical vector components.

$$\begin{aligned}&-\frac{1}{\rho}\nabla p = -\frac{1}{\rho} p_{,}^{j} \mathbf{g}_j = -\frac{1}{\rho}(p_{,}^{j} h_j)\mathbf{g}_j^* \equiv -\frac{1}{\rho} p_{,}^{*j} \mathbf{g}_j^* = -\frac{1}{\rho} p_{,}^{*k} \mathbf{g}_k^*; \\ &-\frac{1}{\rho}\nabla p = -\frac{1}{\rho} p_{,i} \mathbf{g}^i = -\frac{1}{\rho}(p_{,i} g^{ik} h_k)\mathbf{g}_k^* \equiv -\frac{1}{\rho} p_{,}^{*k} \mathbf{g}_k^*\end{aligned} \tag{4.117}$$

Using Eq. (B.3), the physical covariant stress tensor components π_{ij}^* in the unitary contravariant basis can be computed as

$$\begin{aligned}\mathbf{\Pi} &= \pi_{ij}\mathbf{g}^i\mathbf{g}^j = \pi_{ij}(g^{ik}\mathbf{g}_k)(g^{jl}\mathbf{g}_l) \\ &= (\pi_{ij}g^{ik}g^{jl}h_k h_l)\mathbf{g}_k^*\mathbf{g}_l^* \equiv \pi_{ij}^*\mathbf{g}_k^*\mathbf{g}_l^* \\ &\Rightarrow \pi_{ij}^* = (g^{ik}g^{jl}h_k h_l)\pi_{ij}\end{aligned} \quad (4.118)$$

Using Eqs. (4.25, 4.118, B.2) and interchanging the index l with k, the friction covariant stress tensor in the RHS of Eq. (4.109) can be rewritten in the unitary contravariant basis with the physical tensor components.

$$\begin{aligned}\frac{\nabla\cdot\mathbf{\Pi}}{\rho} &= \frac{1}{\rho}\pi_{ij}|_k g^{jk}\mathbf{g}^i \\ &= \frac{1}{\rho}\pi_{ij}|_k g^{jk}(g^{li}\mathbf{g}_l) = \frac{1}{\rho}\pi_{ij}|_k g^{jk}g^{li}(h_l\mathbf{g}_l^*) \\ &= \frac{g^{jk}g^{li}}{\rho g^{ik}g^{jl}h_k}\pi_{ij}^*|_k\mathbf{g}_l^* = \frac{g^{jl}g^{ki}}{\rho g^{il}g^{jk}h_l}\pi_{ij}^*|_l\mathbf{g}_k^*\end{aligned} \quad (4.119)$$

The physical contravariant stress tensor components π^{*ij} in the unitary covariant basis can be computed using Eq. (B.2).

$$\begin{aligned}\mathbf{\Pi} &= \pi^{ij}\mathbf{g}_i\mathbf{g}_j = \pi^{ij}(h_i\mathbf{g}_i^*)(h_j\mathbf{g}_j^*) \\ &= \pi^{ij}h_i h_j\mathbf{g}_i^*\mathbf{g}_j^* \equiv \pi^{*ij}\mathbf{g}_i^*\mathbf{g}_j^* \\ &\Rightarrow \pi^{*ij} = h_i h_j\pi^{ij}\end{aligned} \quad (4.120)$$

Using Eqs. (4.24a, 4.120, B.2), the friction contravariant stress tensor in the RHS of Eq. (4.109) can be written in the unitary covariant basis with the physical tensor components.

$$\begin{aligned}\frac{\nabla\cdot\mathbf{\Pi}}{\rho} &= \frac{1}{\rho}\pi^{ij}|_i\mathbf{g}_j = \frac{1}{\rho}\pi^{ij}|_i h_j\mathbf{g}_j^* \\ &= \frac{1}{\rho h_i}\pi^{*ij}|_i\mathbf{g}_j^* = \frac{1}{\rho h_i}\pi^{*ik}|_i\mathbf{g}_k^*\end{aligned} \quad (4.121)$$

Using Eqs. (B.2, B.3), the gravity tensor in the RHS of Eq. (4.109) can be written in the unitary covariant basis with the physical vector components.

$$\begin{aligned}-\nabla(gz) &= -gz_{,}^{j}\mathbf{g}_j = -g(z_{,}^{j}h_j)\mathbf{g}_j^* \equiv -gz_{,}^{*k}\mathbf{g}_k^*; \\ -\nabla(gz) &= -gz_{,i}\mathbf{g}^i = -g(z_{,i}g^{ik}h_k)\mathbf{g}_k^* \equiv -gz_{,}^{*k}\mathbf{g}_k^*\end{aligned} \quad (4.122)$$

where g (=9.81 m/s^2) is the earth gravity.

4.6.3 Energy (Rothalpy) Equation

The energy equation describes the balance of energies in a control volume. They are based on the first law of thermodynamics for an open system.

The specific rothalpy I for a unit volume is defined as

$$\begin{aligned} I &\equiv h + \frac{1}{2}v^2 - uv_u \\ &= c_p T + \frac{1}{2}v^2 - uv_u \end{aligned} \tag{4.123}$$

where h is the specific enthalpy of fluid.

Using the trigonometric calculation with the velocity triangle, the circumferential absolute velocity results (Fig. 4.6)

$$v_u = u + w_u \tag{4.124}$$

Similarly, the absolute velocity can be written using Eq. (4.124).

$$\begin{aligned} v &= v_u + v_m \\ &= (u + w_u) + v_m \\ &\Rightarrow v^2 = (u + w_u)^2 + v_m^2 \\ &= u^2 + w_u^2 + 2uw_u + v_m^2 \end{aligned} \tag{4.125}$$

Substituting Eqs. (4.124, 4.125) into Eq. (4.123), one obtains the specific rothalpy of the turbomachinery.

$$\begin{aligned} I &\equiv h + \frac{1}{2}v^2 - uv_u \\ &= h + \frac{1}{2}(u^2 + w_u^2 + 2uw_u + v_m^2) - u(u + w_u) \\ &= h + \frac{1}{2}(w_u^2 + v_m^2) - \frac{1}{2}u^2 \\ &= h + \frac{1}{2}w^2 - \frac{1}{2}u^2 \\ &= h + \frac{1}{2}\mathbf{w}\cdot\mathbf{w} - \frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 \end{aligned} \tag{4.126}$$

Therefore, the specific rothalpy I (an invariant) becomes

$$I = c_p T + \frac{1}{2}\mathbf{w}\cdot\mathbf{w} - \frac{1}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 \tag{4.127}$$

Using the first law of thermodynamics for an open system, the energy equation can be written as (Chen 2010).

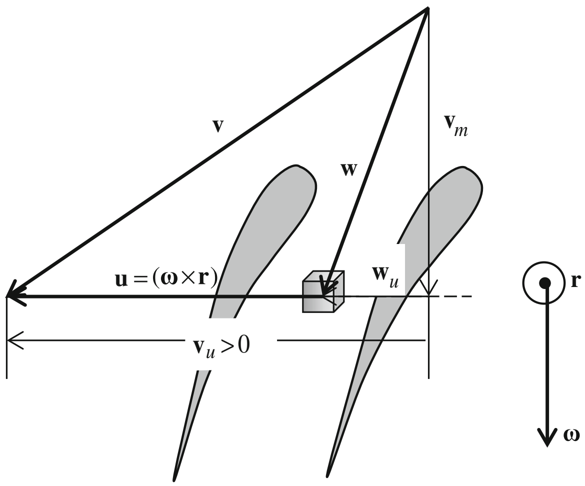

Fig. 4.6 Triangle of velocities in an axial turbomachinery

$$\left(\frac{DI}{Dt}\right)_{\mathrm{rot}} \equiv \frac{\partial I}{\partial t} + \mathbf{w}\cdot\nabla I = \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{1}{\rho}\nabla\cdot\dot{\mathbf{q}} + \frac{1}{\rho}[\mathbf{w}\cdot(\nabla\cdot\mathbf{\Pi}) + (\mathbf{\Pi}\cdot\mathbf{E})] \quad (4.128)$$

in which the first term in the RHS of Eq. (4.128) denotes the specific pressure power, the second term is the specific heat transfer power, and both last terms are the induced specific viscous power of the stress and strain tensors **Π** and **E** of fluid acting upon the control volume.

In the following section, the relating terms in the energy Eq. (4.128) are calculated. Firstly, the substantial time derivative of the specific rothalpy I with respect to the rotating frame can be written in the tensor equation.

$$\begin{aligned}\left(\frac{DI}{Dt}\right)_{\mathrm{rot}} &\equiv \frac{\partial I}{\partial t} + \mathbf{w}\cdot\nabla I \\ &= \dot{I} + w^j I_{,i}\mathbf{g}_j\cdot\mathbf{g}^i \\ &= \dot{I} + w^j I_{,i}\delta_j^i \\ &= \dot{I} + w^i I_{,i}\end{aligned} \quad (4.129a)$$

Using Eq. (4.100), the tensor Eq. (4.129a) can be written in the velocity physical component w^{*i}.

$$\frac{\partial I}{\partial t} + \mathbf{w}\cdot\nabla I = \dot{I} + \frac{w^{*i}I_{,i}}{h_i} \quad (4.129b)$$

The specific pressure power is given in

$$\frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{\dot{p}}{\rho} \quad (4.130)$$

The heat transfer specific power due to heat conduction through the control volume can be written using the Laplacian of fluid temperature T, as given in Eq. (4.37).

$$\begin{aligned} \dot{\mathbf{q}} &= \lambda \nabla T \Rightarrow \\ \frac{1}{\rho}\nabla \cdot \dot{\mathbf{q}} &= \frac{1}{\rho}\nabla \cdot (\lambda \nabla T) = \frac{\lambda}{\rho}\nabla^2 T \\ &= \frac{\lambda}{\rho} g^{ij}\left(T_{,ij} - T_{,k}\Gamma^k_{ij}\right) \end{aligned} \tag{4.131}$$

where λ is the constant thermal conductivity.

The fluid viscous power on the control volume surface can be written in the tensor equation with the stress and strain tensor components of π^{ij} and ε_{ij}.

$$\frac{1}{\rho}[\mathbf{w} \cdot (\nabla \cdot \mathbf{\Pi}) + (\mathbf{\Pi} \cdot \mathbf{E})] = \frac{1}{\rho}\left(w_j \pi^{ij}|_i + \pi^{ij}\varepsilon_{ij}\right) = \frac{1}{\rho}\left(g_{jk} w^k \pi^{ij}|_i + \pi^{ij}\varepsilon_{ij}\right) \tag{4.132a}$$

Using Eqs. (4.100, 4.118), Eq. (4.132a) can be expressed in the physical vector components w^*.

$$\frac{1}{\rho}[\mathbf{w} \cdot (\nabla \cdot \mathbf{\Pi}) + (\mathbf{\Pi} \cdot \mathbf{E})] = \frac{1}{\rho h_i h_j h_k}\left(g_{jk} w^{*k} \pi^{*ij}\big|_i + \frac{\pi^{*ij}\varepsilon^*_{ij}}{g^{ik} g^{jl} h_l}\right). \tag{4.132b}$$

4.7 Basic Equations of Continuum Mechanics

Continuum mechanics deals with the mechanical behavior of continuous materials on the macroscopic scale. The basic equations of continuum mechanics consist of two kinds of equations. First, the equations of conservation of mass, force, and energy can be applied to all materials in any coordinate system. Such equations are the *Cauchy's laws of motion*, which concern with the kinematics of a continuum medium (solids and fluids). Second, the *constitutive equations* describe the macroscopic responses resulting from the internal characteristics of an individual material. The additional constitutive equations help the Cauchy's equations of continuum mechanics in order to predict the responses of the individual material to the applied loads.

Physical laws must always be invariant and independent of the observers from different coordinate systems. Therefore, such equations describing physical laws are generally written in the tensor equations that are always valid for any coordinate system.

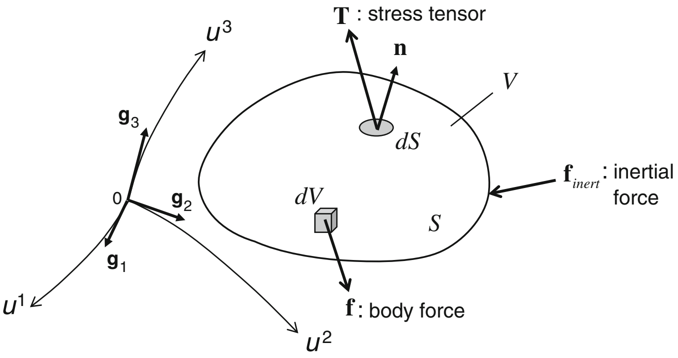

Fig. 4.7 Acting forces on a moving body

4.7.1 Cauchy's Law of Motion

The Cauchy's law of motion is based on the conservation of forces in a continuum medium. Figure 4.7 displays the balance of forces acting on the moving body in a general curvilinear coordinate system u^1, u^2, u^3.

The Cauchy's stress tensor $\mathbf{T}$ can be written in the contravariant second-order tensor as

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j \tag{4.133}$$

The body force $\mathbf{f}$ per volume unit can be expressed in the contravariant first-order tensor (vector).

$$\mathbf{f} = f^j\mathbf{g}_j \tag{4.134}$$

The inertial force acting upon the body can be written in the contravariant first-order tensor (vector)

$$\mathbf{f}_{\text{inert}} = -\rho a^j\mathbf{g}_j = -\rho\ddot{u}^j\mathbf{g}_j \tag{4.135}$$

where ρ is the body density.

Using the D'Alembert principle, one obtains the Cauchy's law of motion in the integral form.

$$\sum\mathbf{F} = \oint_S \mathbf{T}\cdot\mathbf{n}\mathrm{d}S + \int_V \mathbf{f}\mathrm{d}V + \int_V \mathbf{f}_{\text{inert}}\mathrm{d}V = \mathbf{0} \tag{4.136}$$

Applying the Gauss divergence theorem to Eq. (4.136), the first integral over surface S can be transformed into the integral over volume V.

$$\oint_S \mathbf{T} \cdot \mathbf{n} dS = \int_V \nabla \cdot \mathbf{T} dV = \int_V T^{ij}|_i \mathbf{g}_j dV \tag{4.137}$$

where the divergence $\nabla \cdot \mathbf{T}$ of a second-order tensor $\mathbf{T}$ results from Eq. (4.24a)

$$\begin{aligned}\nabla \cdot \mathbf{T} &= T^{ij}|_i \mathbf{g}_j \\ &= \left(T^{ij}_{,i} + \Gamma^i_{im} T^{mj} + \Gamma^j_{im} T^{im} \right) \mathbf{g}_j\end{aligned} \tag{4.138}$$

Applying the coordinate transformation in Eqs. (4.89), (4.137) can be written in the curvilinear coordinates using the Jacobian J.

$$\int_V \nabla \cdot \mathbf{T} dV = \int_V T^{ij}|_i \mathbf{g}_j J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 \tag{4.139}$$

Thus, the conservation of forces can be rewritten in the curvilinear coordinates.

$$\int_V (T^{ij}|_i + f^j - \rho \ddot{u}^j) \mathbf{g}_j J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 = \mathbf{0} \tag{4.140}$$

The Cauchy's tensor equation for an arbitrary volume V becomes

$$T^{ij}|_i + f^j = \rho \ddot{u}^j \tag{4.141}$$

In the case of equilibrium, the Cauchy's tensor equation becomes

$$T^{ij}|_i + f^j = 0 \tag{4.142}$$

The Cauchy stress tensor $\mathbf{T}$ contains nine tensor components:

$$T^{ij} = \tau^{ij} \quad \text{for} \quad i,j = 1,2,3 \tag{4.143}$$

Using Eq. (4.143), one obtains the Cauchy's tensor equations for both cases:

$$\begin{aligned}\tau^{ij}|_i + f^j &= \rho \ddot{u}^j; \\ \tau^{ij}|_i + f^j &= 0\end{aligned} \tag{4.144}$$

within using the Christoffel symbols of second kind, the covariant derivative of the stress tensor components with respect to u^i is defined as

$$\tau^{ij}|_i = \tau^{ij}_{,i} + \Gamma^i_{im} \tau^{mj} + \Gamma^j_{im} \tau^{im} \tag{4.145}$$

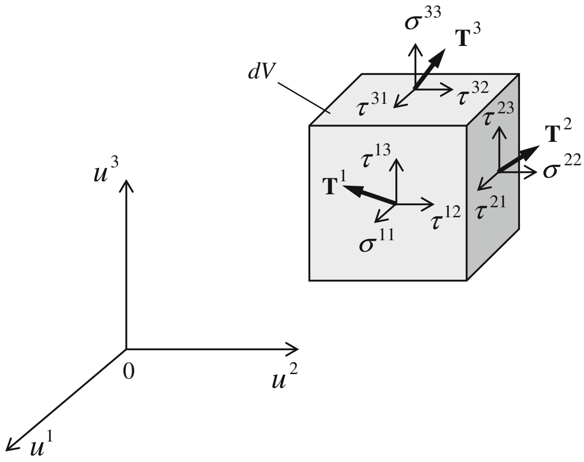

Fig. 4.8 Stress tensor components $\mathbf{T}^i$ acting upon a volume element

The *Cauchy's stress tensor* **T** can be written in a three-dimensional space as

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}^1 \\ \mathbf{T}^2 \\ \mathbf{T}^3 \end{bmatrix} = \begin{pmatrix} \sigma^{11} & \tau^{12} & \tau^{13} \\ \tau^{21} & \sigma^{22} & \tau^{23} \\ \tau^{31} & \tau^{32} & \sigma^{33} \end{pmatrix} \equiv (\tau^{ij}) \tag{4.146}$$

where σ is the normal stress; τ the shear stress.

The stress tensor $\mathbf{T}^i$ on the surface S is defined as the acting force $\boldsymbol{F}$ per unit of surface area.

$$\mathbf{T}^i = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}}{\Delta S} = \frac{\partial \mathbf{F}}{\partial S} \tag{4.147}$$

At each point in the surface S, there are a set of three stress tensor components, one normal stress component σ (pressure) perpendicular to the surface and two shear stress components τ parallel to the surface, as shown in Fig. 4.8. Note that the tensile normal stress denotes a positive normal stress; the compressive stress, a negative normal stress.

In the following section, it is to prove that the second-order Cauchy's stress tensor is symmetric in a free couple-stress body of nonpolar materials. Note that *polar materials* contain couple stresses.

The conservation of angular momentum of the body in equilibrium (cf. Fig. 4.7) can be written as

$$\sum \mathbf{M} = \oint_S (\mathbf{r} \times \mathbf{T}) dS + \int_V (\mathbf{r} \times \mathbf{f}) dV = \mathbf{0} \tag{4.148}$$

The vector **r** can be written in the covariant basis of curvilinear coordinates:

$$\mathbf{r} = x^j \mathbf{g}_j \tag{4.149}$$

Using the Levi-Civita symbols in Eq. (A.5) cf. Appendix A, the angular momentum equation can be expressed as

$$\oint_S \varepsilon_{ijk} x^j T^k \mathbf{g}_k \mathrm{d}S + \int_V \varepsilon_{ijk} x^j f^k \mathbf{g}_k \mathrm{d}V = \mathbf{0} \tag{4.150}$$

Applying the Gauss divergence theorem and using the transformation of curvilinear coordinates, Eq. (4.150) results in

$$\int_V \left(\varepsilon_{ijk} (x^j T^{mk})\big|_m + \varepsilon_{ijk} x^j f^k \right) \mathbf{g}_k J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 = \mathbf{0} \tag{4.151}$$

Calculating the covariant derivative of the first term in Eq. (4.151), one obtains

$$\int_V \varepsilon_{ijk} \left(x^j\big|_m T^{mk} + x^j T^{mk}\big|_m \right) \mathbf{g}_k J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 + \int_V \left(\varepsilon_{ijk} x^j f^k \right) \mathbf{g}_k J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 = \mathbf{0} \tag{4.152}$$

Rearranging the second and third terms in Eq. (4.152), one obtains

$$\int_V \varepsilon_{ijk} x^j\big|_m T^{mk} \mathbf{g}_k J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 + \int_V \varepsilon_{ijk} x^j \left(T^{mk}\big|_m + f^k \right) \mathbf{g}_k J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 = \mathbf{0} \tag{4.153}$$

Due to the balance of forces in Eq. (4.142), the second integral in Eq. (4.153) equals zero.

Therefore, the tensor equation of angular momentum Eq. (4.153) results in

$$\int_V \varepsilon_{ijk} x^j\big|_m T^{mk} \mathbf{g}_k J \mathrm{d}u^1 \mathrm{d}u^2 \mathrm{d}u^3 = \mathbf{0} \tag{4.154}$$

Using the relation of $x^j\big|_m = \delta^j_m$, one obtains for an arbitrary volume V.

$$\begin{aligned} &\varepsilon_{ijk} x^j\big|_m T^{mk} = \varepsilon_{ijk} \delta^j_m T^{mk} = \varepsilon_{ijk} T^{jk} = 0 \\ &\Rightarrow \varepsilon_{ijk} \tau^{jk} = 0 \end{aligned} \tag{4.155}$$

Note that $\varepsilon_{ikj} = -\varepsilon_{ijk}$, one obtains from Eq. (4.155)

$$\begin{aligned} &\varepsilon_{ijk}\tau^{jk} = \frac{1}{2}\varepsilon_{ijk}\left(\tau^{jk} - \tau^{kj}\right) = 0 \\ &\Rightarrow \tau^{jk} = \tau^{kj} \quad \text{for} \quad j,k = 1,2,3 \end{aligned} \tag{4.156}$$

This result proves that the Cauchy's stress tensor is *symmetric* in the free couple-stress body. In a three-dimensional space, there are six symmetric tensor components of shear stress τ^{ij} and three diagonal tensor components of normal stress σ^{ii}, as shown in Eq. (4.146).

It is obvious, the Cauchy's stress tensor is *non-symmetric* if the couple stresses act on the body.

4.7.2 Principal Stresses of Cauchy's Stress Tensor

The Cauchy's stress tensor **T** in a three-dimensional space generally has three invariants that are independent of any chosen coordinate system. These invariants are the normal stresses in the principal directions perpendicular to the principal planes. In fact, the principal normal stresses are the eigenvalues of the stress tensor, where only the normal stresses act on the principal planes in which the remaining shear stresses equal zero. The eigenvectors related to their eigenvalues have the same directions of the principal directions, as shown in Fig. 4.9.

The principal stress vector $\mathbf{T}^i$ of the stress tensor on the normal unit vector $\mathbf{n}_i$ (parallel to the principal direction) can be expressed as

$$\mathbf{T}^i = \lambda \mathbf{n}_i \tag{4.157}$$

Furthermore, the principal stress vector can be written in a linear form of the stress tensor components and their relating normal unit vectors.

$$\mathbf{T}^i = \sigma^{ij}\mathbf{n}_j \quad \text{for} \quad j = 1,2,3 \tag{4.158}$$

where using the Kronecker delta, the normal unit vector $\mathbf{n}_j$ is defined as

$$\mathbf{n}_i = \mathbf{n}_j \delta_i^j \quad \text{for} \quad i,j = 1,2,3 \tag{4.159}$$

The relation between the unit vectors in Eq. (4.159) denotes that all shear stresses vanish in the principal planes perpendicular to the unit vectors $\mathbf{n}_j$ with all indices $j \neq i$. In the case of $j \equiv i$, only the principal stresses act on the principal planes.

Substituting Eqs. (4.158, 4.159) into Eq. (4.157), one obtains

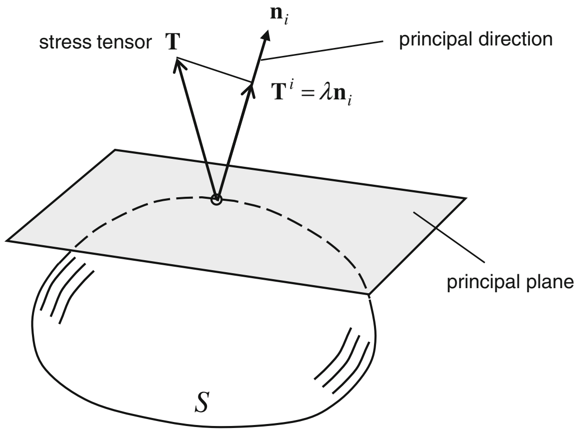

Fig. 4.9 Acting principal stress $\mathbf{T}^i$ on a principal plane

$$(\sigma^{ij} - \lambda\delta_i^j)\mathbf{n}_j = \mathbf{0} \quad \text{for} \quad i,j = 1,2,3 \tag{4.160}$$

For non-trivial solutions of Eq. (4.160), its determinant must equal zero.

Therefore,

$$\det(\sigma^{ij} - \lambda\delta_i^j) = \begin{vmatrix} (\sigma^{11} - \lambda) & \sigma^{12} & \sigma^{13} \\ \sigma^{21} & (\sigma^{22} - \lambda) & \sigma^{23} \\ \sigma^{31} & \sigma^{32} & (\sigma^{33} - \lambda) \end{vmatrix} = 0 \tag{4.161}$$

This equation is called the *characteristic equation* of the eigenvalues of the second-order stress tensor **T**.

Calculating the determinant of Eq. (4.161), the third-order characteristic equation of λ results in

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0 \tag{4.162}$$

within

$$\begin{cases} I_1 = Tr(\sigma^{ii}); \\ I_2 = \frac{1}{2}(\sigma^{ii}\sigma^{jj} - \sigma^{ij}\sigma^{ji}); \\ I_3 = \det(\sigma^{ij}) \end{cases} \tag{4.163}$$

Generally, there are three real roots of Eq. (4.162) for the eigenvalues in the principal directions.

$$\begin{cases} \lambda_1 = \sigma^1 \equiv \sigma^{\max} \\ \lambda_2 = I_1 - \sigma^1 - \sigma^3 = \sigma^2 \\ \lambda_3 = \sigma^3 \equiv \sigma^{\min} \end{cases} \tag{4.164}$$

The eigenvalues of Eq. (4.164) are the roots of the *Cayley–Hamilton theorem*. The stress tensor **T** can be written in the principal directions as follows:

$$\mathbf{T} \equiv (\sigma^{ij}) = \begin{pmatrix} \sigma^1 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^3 \end{pmatrix} \tag{4.165}$$

The maximum shear stress occurs at the angle of 45° between the smallest and largest principal stress planes with the value

$$\tau^{\max} = \frac{\left|\sigma^{\max} - \sigma^{\min}\right|}{2} = \frac{\left|\sigma^1 - \sigma^3\right|}{2} \text{ at } \sigma^{\text{middle}} = \frac{\left|\sigma^1 + \sigma^3\right|}{2}. \tag{4.166}$$

4.7.3 Cauchy's Strain Tensor

The *Cauchy's strain tensor* describes the infinitesimal deformation of a solid body in which the displacement between two arbitrary points in the material is much smaller than any relevant dimension of the body. In this case, the Cauchy's strain tensor is based on the small deformation theory or linear deformation theory.

The vector $\mathbf{R}(u^1, u^2, u^3, t)$ of an arbitrary point P of the body at the time t after deformation results from the vector $\mathbf{r}(u^1, u^2, u^3)$ of the same point at the time t_0 before deformation and the small deformation vector $\mathbf{v}(u^1, u^2, u^3, t)$, as shown in Fig. 4.10.

$$\mathbf{R}(u^1, u^2, u^3, t) = \mathbf{r}(u^1, u^2, u^3) + \mathbf{v}(u^1, u^2, u^3, t) \tag{4.167}$$

The covariant basis vectors of both curvilinear coordinates at the times t_0 and t result from

$$\frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial \mathbf{r}}{\partial u^i} + \frac{\partial \mathbf{v}}{\partial u^i} \quad \Leftrightarrow \quad \mathbf{G}_i = \mathbf{g}_i + \mathbf{v}_{,i} \tag{4.168}$$

The difference of two segments can be calculated by

$$\begin{aligned} \mathrm{d}\mathbf{R}^2 - \mathrm{d}\mathbf{r}^2 &= \mathbf{G}_i \mathbf{G}_j \mathrm{d}u^i \mathrm{d}u^j - \mathbf{g}_i \mathbf{g}_j \mathrm{d}u^i \mathrm{d}u^j \\ &= (G_{ij} - g_{ij}) \mathrm{d}u^i \mathrm{d}u^j \equiv 2\gamma_{ij} \mathrm{d}u^i \mathrm{d}u^j \end{aligned} \tag{4.169}$$

where γ_{ij} are the components of the second-order strain tensor.

Calculating the covariant metric coefficients, one obtains

$$\begin{aligned} &\mathbf{G}_i \cdot \mathbf{G}_j = (\mathbf{g}_i + \mathbf{v}_{,i}) \cdot (\mathbf{g}_j + \mathbf{v}_{,j}) = \mathbf{g}_i \cdot \mathbf{g}_j + \mathbf{g}_i \mathbf{v}_{,j} + \mathbf{g}_j \mathbf{v}_{,i} + \mathbf{v}_{,i} \mathbf{v}_{,j} \\ &\Rightarrow G_{ij} - g_{ij} = \mathbf{g}_i \cdot \mathbf{v}_{,j} + \mathbf{g}_j \cdot \mathbf{v}_{,i} + \mathbf{v}_{,i} \cdot \mathbf{v}_{,j} \end{aligned} \tag{4.170}$$

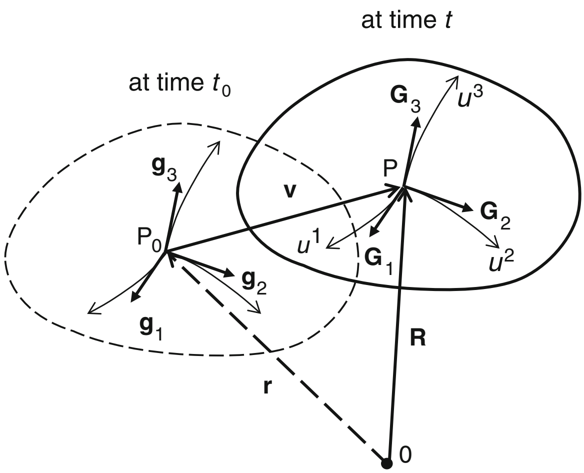

Fig. 4.10 Infinitesimal deformation of a solid body

Thus, the stress tensor components can be calculated as

$$\begin{aligned}\gamma_{ij} &= \frac{1}{2}(G_{ij} - g_{ij}) = \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{v}_{,j} + \mathbf{g}_j \cdot \mathbf{v}_{,i} + \mathbf{v}_{,i} \cdot \mathbf{v}_{,j}) \\ &= \frac{1}{2}\left(\mathbf{g}_i \cdot \frac{\partial \mathbf{v}}{\partial u^j} + \mathbf{g}_j \cdot \frac{\partial \mathbf{v}}{\partial u^i} + \frac{\partial \mathbf{v}}{\partial u^i} \cdot \frac{\partial \mathbf{v}}{\partial u^j}\right)\end{aligned} \tag{4.171}$$

The deformation vector **v** can be written in the contravariant and covariant bases.

$$\mathbf{v} = v_k \mathbf{g}^k = v^k \mathbf{g}_k \tag{4.172}$$

Using Eqs. (2.207, 2.200), one obtains the covariant partial derivatives

$$\begin{aligned}\mathbf{v}_{,i} &= v_k|_i\, \mathbf{g}^k; \\ \mathbf{v}_{,j} &= v^k\big|_j\, \mathbf{g}_k\end{aligned} \tag{4.173}$$

within the covariant derivatives result from Eqs. (2.208, 2.201) in

$$\begin{aligned}v_k|_i &= v_{k,i} - \Gamma^j_{ki} v_j; \\ v^k\big|_j &= v^k_{,j} + \Gamma^k_{ji} v^i\end{aligned} \tag{4.174}$$

Substituting Eq. (4.173) into Eq. (4.171), the strain tensor components of the Cauchy's strain tensor $\boldsymbol{\gamma}$ can be rewritten as

$$\begin{aligned}\gamma_{ij} &= \frac{1}{2}(\mathbf{g}_i \cdot \mathbf{v}_{,j} + \mathbf{g}_j \cdot \mathbf{v}_{,i} + \mathbf{v}_{,i} \cdot \mathbf{v}_{,j}) \\ &= \frac{1}{2}(v_k|_j \mathbf{g}^k \cdot \mathbf{g}_i + v_k|_i \mathbf{g}^k \cdot \mathbf{g}_j + v_l|_i v^k\big|_j \mathbf{g}_k \cdot \mathbf{g}^l) \\ &= \frac{1}{2}(v_k|_j \delta^k_i + v_k|_i \delta^k_j + v_l|_i v^k\big|_j \delta^l_k) \\ &= \frac{1}{2}(v_i|_j + v_j\big|_i + v_k|_i \cdot v^k\big|_j)\end{aligned} \tag{4.175}$$

The third term on the RHS of Eq. (4.175) is very small in the infinitesimal deformation. Therefore, the components of the Cauchy's strain tensor $\boldsymbol{\gamma}$ become

$$\gamma_{ij} \approx \frac{1}{2}\left(v_i\big|_j + v_j\big|_i\right) \tag{4.176}$$

Substituting Eq. (4.174) into Eq. (4.176) and using the symmetry of the Christoffel symbols, the Cauchy's tensor component can be written in the linear elasticity.

$$\begin{aligned}\gamma_{\mathrm{ij}} &= \frac{1}{2}(v_{i,j} + v_{j,i}) - \Gamma_{ij}^{k} v_k \\ &= \frac{1}{2}(v_{j,i} + v_{i,j}) - \Gamma_{ji}^{k} v_k \\ &= \gamma_{ji} (\text{q.e.d.})\end{aligned} \tag{4.177}$$

This result proves that the Cauchy's strain tensor $\boldsymbol{\gamma}$ is also symmetric.

As an example, the *Cauchy's strain tensor* $\boldsymbol{\gamma}$ can be written in a three-dimensional space:

$$\boldsymbol{\gamma} = \begin{pmatrix} \varepsilon_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \varepsilon_{22} & \gamma_{21} \\ \gamma_{31} & \gamma_{32} & \varepsilon_{33} \end{pmatrix} \equiv (\gamma_{ij}) \tag{4.178}$$

where ε_{ii} are the normal strains; γ_{ij} the shear strains of the second-order strain tensor (matrix).

Note that the Christoffel symbols in Eq. (4.177) vanish in Cartesian coordinates (x, y, z) in a three-dimensional space $\mathbf{E}^3$. Therefore, the normal and shear strains of Eq. (4.178) can be computed as follows:

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x};\ \varepsilon_{yy} = \frac{\partial v}{\partial y};\ \varepsilon_{zz} = \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \gamma_{yx} \\ \gamma_{yz} &= \tfrac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \gamma_{zy} \\ \gamma_{xz} &= \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \gamma_{zx} \end{aligned}\right\} \Rightarrow \boldsymbol{\gamma} = \begin{pmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{pmatrix} \tag{4.179}$$

where u, v, and w are the vector components of the deformation vector $\mathbf{v}$.

Similar to the principal stresses, the *principal strains* are invariants that are independent of any chosen coordinate system. In this case, only the principal strains (eigenvalues) exist in the principal directions (eigenvectors) of the principal planes in which the shear strains equal zero.

The characteristic equation of the eigenvalues λ of the strain tensor $\boldsymbol{\gamma}$ results from

$$\begin{aligned} &(\gamma_{ij} - \lambda\delta_i^j)\mathbf{n}_j = \mathbf{0} \quad \text{for} \quad i,j = 1,2,3 \\ &\Rightarrow \det(\gamma_{ij} - \lambda\delta_i^j) = 0 \end{aligned} \tag{4.180}$$

There are three real roots of Eq. (4.180) for the principal strains (eigenvalues) in the principal directions (eigenvectors).

$$\lambda_1 = \varepsilon_1; \quad \lambda_2 = \varepsilon_2; \quad \lambda_3 = \varepsilon_3 \tag{4.181}$$

The Cauchy's strain tensor $\boldsymbol{\gamma}$ can be written in the principal directions:

$$\boldsymbol{\gamma} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \equiv (\varepsilon_{ij}). \tag{4.182}$$

4.7.4 Constitutive Equations of Elasticity Laws

The constitutive equations describe the macroscopic responses resulting from the internal characteristics of an individual material. The linear elasticity law of materials shows the relation between the stress tensor **T** and strain tensor $\boldsymbol{\gamma}$ in general curvilinear coordinates of an N-dimensional space.

In general, the Hooke's law is valid in the linear elasticity of material. As a result, the stress tensor component is proportional to the strain tensor component in the elasticity range of an individual material compared to the spring force that is proportional to its displacement by a spring constant.

The Hooke's law of the linear elasticity law is written as

$$\tau^{ij} = E^{ijkl}\gamma_{kl} \quad \text{for} \quad i, j, k, l = 1, 2, \ldots, N \tag{4.183}$$

where E^{ijkl} is the fourth-order *elasticity tensor* that only depends on the material characteristics. The elasticity tensor has 81 components (=3^4) in a three-dimensional space; N^4 components in an N-dimensional space.

The elasticity tensor is a function of the elasticity modulus E (Young's modulus) and the Poisson's ratio v (Green and Zerna 1968).

$$E^{ijkl} = \mu\left(g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2v}{(1-2v)}g^{ij}g^{kl}\right) \tag{4.184}$$

where μ is the shear modulus (modulus of rigidity) that can be defined as

$$\mu = \frac{E}{2(1+v)} \tag{4.185}$$

Some moduli of steels are mostly used in engineering applications:

- $E \approx 212$ GPa (low-alloy steels); 230 GPa (high-alloy steels);
- $\mu \approx 0.385E \ldots 0.400E$;
- $v \approx 0.25 \ldots 0.30$ (most metals).

According to the Hooke's law, the elasticity equation results from Eqs. (4.183–4.185) for $i, j, k, l = 1,2,\ldots,N$.

$$\tau^{ij} = \mu\left(g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu}{(1-2\nu)}g^{ij}g^{kl}\right)\gamma_{kl} \tag{4.186}$$

The basic tensor equations of the linear elasticity theory comprise Eqs. (4.144), (4.177), and (4.186):

$$\begin{aligned} &\tau^{ij}_{,i} + \Gamma^{i}_{im}\tau^{mj} + \Gamma^{j}_{im}\tau^{im} = \rho\ddot{u}^{j} - f^{j} \\ &\gamma_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) - \Gamma^{k}_{ij}v_{k} \\ &\tau^{ij} = \mu\left(g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu}{(1-2\nu)}g^{ij}g^{kl}\right)\gamma_{kl}. \end{aligned} \tag{4.187}$$

4.8 Maxwell's Equations of Electrodynamics

Maxwell's equations are the fundamental equations for electrodynamics, telecommunication technologies, quantum electrodynamics, and special and general relativity theories. These space–time equations describe mutual interactions between the charges, currents, electric, and magnetic fields in a matter.

The Maxwell's equations of electromagnetism are a system of four inhomogeneous partial differential equations in four space–time dimensions (x, y, z, t) of the electric field strength **E**, magnetic field density **B**, electric displacement **D**, and magnetic field strength **H** (Griffiths 1999; Lawden 2002).

The Maxwell's equations in a matter can be written in integral, differential, and tensor equations using Eqs. (4.31, 4.33) and SI units. The tensor equations with physical components are formulated in an orthogonal coordinate system.

- **Gauss's law for electric fields**:

$$\oint_S \mathbf{D}\cdot\mathbf{n}\,dA = q_{\text{enc}} \text{ (Integral equation)} \tag{4.188a}$$

$$\nabla\cdot\mathbf{D} = \rho \text{ (Differential equation)} \tag{4.188b}$$

$$\frac{1}{J}\frac{\partial}{\partial u^i}\left(JD^i\right) = \rho \text{ (Tensor equation)} \tag{4.188c}$$

$$\frac{1}{J}\frac{\partial}{\partial u^i}\left(J\frac{D^{*i}}{h_i}\right) = \rho \text{ (Tensor equation with physical components)} \tag{4.188d}$$

- **Gauss's law for magnetic fields**:

$$\oint_S \mathbf{B} \cdot \mathbf{n} dA = 0 \text{ (Integral equation)} \tag{4.189a}$$

$$\nabla \cdot \mathbf{B} = 0 \text{ (Differential equation)} \tag{4.189b}$$

$$\frac{1}{J} \frac{\partial}{\partial u^i} \left(J B^i \right) = 0 \text{ (Tensor equation)} \tag{4.189c}$$

$$\frac{1}{J} \frac{\partial}{\partial u^i} \left(J \frac{B^{*i}}{h_i} \right) = 0 \text{ (Tensor equation with physical comp.)} \tag{4.189d}$$

- **Faraday's law**:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} dA \text{ (Integral equation)} \tag{4.190a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ (Differential equation)} \tag{4.190b}$$

$$\hat{\varepsilon}^{ijk} E_{k,j} = -\frac{\partial B^i}{\partial t} \text{ (Tensor equation)} \tag{4.190c}$$

$$\frac{1}{J} \left(h_k E^*_{k,j} - h_j E^*_{j,k} \right) = -\frac{1}{h_i} \frac{\partial B^{*i}}{\partial t} \text{ (Tensor equation with physical comp.)} \tag{4.190d}$$

- **Ampere–Maxwell law**:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{\text{enc}} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{n} dA \text{ (Integral equation)} \tag{4.191a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \text{ (Differential equation)} \tag{4.191b}$$

$$\hat{\varepsilon}^{ijk} H_{k,j} = J^i + \frac{\partial D^i}{\partial t} \text{ (Tensor equation)} \tag{4.191c}$$

$$\frac{1}{J} \left(h_k H^*_{k,j} - h_j H^*_{j,k} \right) = \frac{J^{*i}}{h_i} + \frac{1}{h_i} \frac{\partial D^{*i}}{\partial t} \text{ (Tensor eq. with physical comp.)} \tag{4.191d}$$

where

n is the normal unit vector to the surface S;

q_{enc} (**r**, t) is the four-dimensional space–time-enclosed electric charge;

ρ (**r**, t)	is the four-dimensional space–time electric charge volumetric density;
I_{enc} (**r**, t)	is the four-dimensional space–time-enclosed electric current:
J (**r**, t)	is the four-dimensional space–time electric current volumetric density;
J^*	is the physical component of **J**;
J	is the Jacobian;
h_i	is the norm of the covariant basis $\mathbf{g}_i$.

Note that the Maxwell's equations are invariant at the Lorentz transformation (Landau and Lifshitz 1962). However, they will be studied in the four-dimensional space–time manifold by the Poincaré transformation.

The electric displacement **D** is related to the electric field strength **E** by the matter permittivity ε.

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{4.192}$$

in which $\varepsilon = 1/(\mu c^2)$ is the matter permittivity; $c \approx 3 \times 10^8$ m/s is the light speed in vacuum; μ is the matter permeability ($\mu_0 = 4\pi \times 10^{-7}$ N/A^2 for vacuum).

Analogously, the relation between the magnetic field strength **H** and magnetic field density **B** can be written as

$$\mathbf{B} = \mu \mathbf{H} \tag{4.193}$$

where μ is the matter permeability.

Taking curl ($\nabla\times$) of the curl Eqs. (4.190a, 4.191a) and using the curl identity, as given in Eq. (C.23), the homogenous wave equations for the electric and magnetic field strengths of **E** and **H** in vacuum (i.e., without any source exists, $\mathbf{J} = \rho = 0$) can be derived in

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - c^2 \nabla^2 \mathbf{E} = \mathbf{0} \tag{4.194}$$

$$\frac{\partial^2 \mathbf{H}}{\partial t^2} - c^2 \nabla^2 \mathbf{H} = \mathbf{0} \tag{4.195}$$

where c is the propagating speed of the electromagnetic waves in vacuum, which equals the light speed.

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.998 \times 10^8 \, \mathrm{m\,s^{-1}} \tag{4.196}$$

with $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 (H/m); $\varepsilon_0 = 8.85 \times 10^{-12}$ C^2/(N m^2).

Therefore, Maxwell postulated that light is induced by the electromagnetic disturbance propagated through the field according to the electromagnetic laws. Furthermore, the law of conservation of the four-current density vector **J** is similar to the continuity equation of fluid dynamics, as given in Eq. (4.93).

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r},t) = 0 \tag{4.197}$$

In the following section, the Maxwell's equations can be expressed in the space–time manifold in which the Einstein special relativity theory has been usually formulated. The space–time coordinates can be defined by Poincaré in the Minkowski space where three real space dimensions in Euclidean space of x, y, and z are combined with a single time dimension t to generate a four-dimensional space–time manifold with a fourth imaginary dimension of $x^4 = jct$ in the Poincaré group (Lawden 2002; Landau and Lifshitz 1962). The coordinates of this four-dimensional space-time are called the pseudo-Euclidean coordinates that can be written as

$$x^1 = x; \quad x^2 = y; \quad x^3 = z; \quad x^4 = \sqrt{-1}ct \equiv jct \tag{4.198}$$

Thus, the skew-symmetric tensor components F_{kl} and G_{kl} of the electromagnetic fields **B** and **H** are expressed in the pseudo-Euclidean coordinates for $k, l = 1, 2, 3, 4$.

$$\begin{cases} F_{12} = B_z; F_{23} = B_x; F_{31} = B_y; F_{4k} = jE_k/c \\ G_{12} = H_z; G_{23} = H_x; G_{31} = H_y; G_{4k} = jcD_k \end{cases} \tag{4.199}$$

Similarly, the four-current density vector **J** can be defined as

$$\mathbf{J} \equiv \begin{cases} J_1 = J_x \\ J_2 = J_y \\ J_3 = J_z \\ J_4 = jc\rho = \sqrt{-1}c\rho \end{cases} \tag{4.200}$$

According to Eqs. (4.192, 4.193, 4.196), the relation between G_{kl} and F_{kl} in Eq. (4.199) for vacuum results in

$$G_{kl} = \frac{F_{kl}}{\mu_0} \tag{4.201}$$

Therefore, the first and fourth inhomogeneous Maxwell's Eqs. (4.188a, 4.191a) can be expressed in the tensor equation of G_{kl}.

$$\begin{cases} \dfrac{\partial G_{kl}}{\partial x^l} \equiv G_{kl,l} = J_k \Rightarrow \\ \dfrac{\partial F_{kl}}{\partial x^l} \equiv F_{kl,l} = \mu_0 J_k \quad \text{for} \quad k, l = 1, 2, 3, 4 \end{cases} \tag{4.202}$$

Thus, Eq. (4.202) can be written in the four-dimensional space–time coordinates:

$$\begin{cases} \frac{\partial F_{12}}{\partial x^2} + \frac{\partial F_{13}}{\partial x^3} + \frac{\partial F_{14}}{\partial x^4} = \mu_0 J_1 \\ \frac{\partial F_{21}}{\partial x^1} + \frac{\partial F_{23}}{\partial x^3} + \frac{\partial F_{24}}{\partial x^4} = \mu_0 J_2 \\ \frac{\partial F_{31}}{\partial x^1} + \frac{\partial F_{32}}{\partial x^2} + \frac{\partial F_{34}}{\partial x^4} = \mu_0 J_3 \\ \frac{\partial F_{41}}{\partial x^1} + \frac{\partial F_{42}}{\partial x^2} + \frac{\partial F_{43}}{\partial x^3} = \mu_0 J_4 \end{cases} \tag{4.203}$$

Similarly, the second and third homogenous Maxwell's Eqs. (4.189a, 4.190a) can be written in the tensor equation of F_{kl}.

$$\begin{cases} \frac{\partial F_{kl}}{\partial x^m} + \frac{\partial F_{lm}}{\partial x^k} + \frac{\partial F_{mk}}{\partial x^l} \equiv \\ F_{kl,m} + F_{lm,k} + F_{mk,l} = 0 \quad \text{for} \quad k, l, m = 1, 2, 3, 4 \end{cases} \tag{4.204}$$

where the covariant electromagnetic field tensors can be defined by Lawden (2002).

$$F_{ij} \equiv \begin{pmatrix} 0 & B_z & -B_y & -jE_x/c \\ -B_z & 0 & B_x & -jE_y/c \\ B_y & -B_x & 0 & -jE_z/c \\ jE_x/c & jE_y/c & jE_z/c & 0 \end{pmatrix} \tag{4.205}$$

Equation (4.204) can be written in the four-dimensional space–time coordinates:

$$\begin{cases} \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^3} = 0 \\ \frac{\partial F_{34}}{\partial x^2} + \frac{\partial F_{42}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^4} = 0 \\ \frac{\partial F_{41}}{\partial x^3} + \frac{\partial F_{13}}{\partial x^4} + \frac{\partial F_{34}}{\partial x^1} = 0 \\ \frac{\partial F_{12}}{\partial x^4} + \frac{\partial F_{24}}{\partial x^1} + \frac{\partial F_{41}}{\partial x^2} = 0 \end{cases} \tag{4.206}$$

The potential vector A(**r**, *t*) can be defined as a potential scalar ϕ.

$$\mathbf{A}(\mathbf{r}, t) = \begin{cases} A_1 = A_x \\ A_2 = A_y \\ A_3 = A_z \\ A_4 = j\phi(\mathbf{r}, t)/c \end{cases} \tag{4.207}$$

The skew-symmetric electromagnetic field tensor component F_{kl} can be written in the potential vector components.

$$\begin{aligned} F_{kl} &= \frac{\partial A_k}{\partial x^l} - \frac{\partial A_l}{\partial x^k} \\ &= A_{k,l} - A_{l,k} \quad \text{for} \quad k, l = 1, 2, 3, 4 \end{aligned} \tag{4.208}$$

Substituting Eqs. (4.202, 4.204, 4.208), the Maxwell's equations can be expressed in the potential vector **A**.

$$\begin{aligned} \frac{\partial^2 A_k}{(\partial x^l)^2} - \frac{\partial^2 A_l}{\partial x^k \partial x^l} &\equiv A_{k,ll} - A_{l,kl} \\ &= -\mu_0 J_k \quad \text{for} \quad k, l = 1, 2, 3, 4 \end{aligned} \tag{4.209}$$

Using Eqs. (4.199, 4.208), the magnetic field strength **H** can be rewritten in the potential vector **A**.

$$\mathbf{B} = \mu \mathbf{H} = \nabla \times \mathbf{A} \tag{4.210}$$

Using Faraday's law, the electric field strength **E** can be expressed in the potential vector **A** and the potential scalar ϕ.

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{4.211}$$

In the Lorenz gauge, the relation between the potential scalar and vector can be written as (Lawden 2002).

$$\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \tag{4.212}$$

Using Maxwell's equations and product rules of vector calculus, the wave equations of the potential scalar and vector result in

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{4.213}$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \tag{4.214}$$

4.9 Einstein Field Equations

Einstein field equations (EFE) in vacuum are the fundamental equations in the general relativity theory. They describe the interactions between the gravitational field, physical characteristics of matters, and energy momentum tensor (Landau and Lifshitz 1962). All tensors using in the general relativity theory have been mostly written in the abstract index notation defined by Penrose (2005). This index notation uses the indices to express the tensor types rather than their covariant components in the basis $\{\mathbf{g}^i\}$

According to Eqs. (2.250a, 2.251) and using the tensor contraction laws, the Einstein tensor can be written in the covariant tensor components.

$$\begin{aligned} G_{ij} &= g_{ik} G_j^k \\ &= g_{ik}\left(R_j^k - \frac{1}{2}\delta_j^k R\right) \\ &= R_{ij} - \frac{1}{2} g_{ij} R \end{aligned}$$

Therefore, the Einstein field equations can be expressed as

$$G_{ij} \equiv \left(R_{ij} - \frac{1}{2} g_{ij} R\right) = -\frac{8\pi G}{c^4} T_{ij} \tag{4.215}$$

where

G_{ij} is the covariant Einstein tensor in Eq. (2.251);
R_{ij} is the Ricci tensor in Eq. (2.242);
g_{ij} is the covariant metric tensor components;
R is the Ricci curvature tensor in Eq. (2.248);
G is the universal gravitational constant (= 6.673×10^{-11} N m^2/kg^2);
c is the light speed ($\approx 3 \times 10^8$ m/s);
T_{ij} is the kinetic energy-momentum tensor.

The kinetic energy-momentum tensor in vacuum can be calculated from the cosmological constant Λ and the covariant metric tensor components g_{ij}.

$$T_{ij} = \frac{-\Lambda c^4}{8\pi G} g_{ij} \tag{4.216}$$

in which the cosmological constant is equivalent to an energy density in a vacuum space. According to a recent measurement, the cosmological constant Λ is on the order of 10^{-52} m^{-2} and proportional to the dark-energy density ρ with a factor of $8\pi G$ used in the general relativity.

$$\Lambda = 8\pi G\rho \tag{4.217}$$

In the case of a positive energy density of the vacuum space ($\Lambda > 0$), the related negative pressure will cause an accelerating expansion of the universe.

In relativity electromagnetism, the kinetic energy-momentum tensor T_{ij} can be calculated from the energy-momentum tensor of the electromagnetic field S_{ij} (electromagnetic stress-energy tensor) according to (Lawden 2002).

$$-T_{ij} = S_{ij} = \frac{1}{\mu_0}\left(F_{ik}F_{jk} - \frac{1}{4}\delta_{ij}F_{kl}F_{kl}\right) \tag{4.218}$$

in which δ_{ij} is the Kronecker delta.

Substituting Eq. (4.218) into Eq. (4.215), the Einstein–Maxwell equations can be generally written with the cosmological constant Λ.

$$\begin{aligned} G_{ij} - g_{ij}\Lambda &\equiv \left(R_{ij} - \frac{1}{2}g_{ij}R\right) - g_{ij}\Lambda \\ &= R_{ij} - \left(\frac{1}{2}R + \Lambda\right)g_{ij} \\ &= -\frac{8\pi G}{c^4}T_{ij} \\ &= \frac{8\pi G}{c^4\mu_0}\left(F_{ik}F_{jk} - \frac{1}{4}\delta_{ij}F_{kl}F_{kl}\right) \end{aligned} \tag{4.219}$$

where R is the Ricci curvature tensor can be obtained according to Eqs. (2.248, 2.249).

$$\begin{aligned} R &= R_{ij}g^{ij} \\ &= \left(\frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \frac{1}{J}\frac{\partial(J\Gamma_{ij}^m)}{\partial u^m} + \Gamma_{in}^m\Gamma_{jm}^n\right)g^{ij} \end{aligned} \tag{4.220}$$

The Ricci curvature tensor can be expressed in the kinetic energy-momentum tensor T (Cahill 2013):

$$R = \frac{8\pi G}{c^4}T_i^i = \frac{8\pi G}{c^4}T \tag{4.221}$$

The covariant electromagnetic field tensors F_{ij} in Eq. (4.219) are given according to Eq. (4.205).

$$F_{ij} = \begin{pmatrix} 0 & B_z & -B_y & -jE_x/c \\ -B_z & 0 & B_x & -jE_y/c \\ B_y & -B_x & 0 & -jE_z/c \\ jE_x/c & jE_y/c & jE_z/c & 0 \end{pmatrix} \tag{4.222}$$

where E is the electric field strength; B is the magnetic field density; c is the light speed in vacuum.

4.10 Schwarzschild's Solution of the Einstein Field Equations

In the case of ignoring the cosmological constant Λ and the negligibly small energy-momentum tensor ($T_{ij} = 0$) in an empty space of small scales ($R = 0$), the Einstein field equation in Eq. (4.219) can be written in a simple tensor equation as

$$R_{ij} = 0 \tag{4.223}$$

The Schwarzschild's solution of Eq. (4.223) can be derived for a spherically symmetric empty space with spherical coordinates (r, ϕ, θ) outside an object with a mass M (Cahill 2013; Susskind and Lindesay 2005).

The purely imaginary distance ds along a curve C in the spherical space can be written as

$$\begin{aligned} \mathrm{d}s^2 &= -c^2\mathrm{d}\tau^2 \\ &= g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \\ &= -g_{00}c^2\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \\ &= -\left(1-\frac{2MG}{c^2r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{2MG}{c^2r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\phi^2 + \sin^2\phi\mathrm{d}\theta^2) \end{aligned} \tag{4.224a}$$

Therefore,

$$\mathrm{d}\tau^2 = \left(1-\frac{2MG}{c^2r}\right)\mathrm{d}t^2 - \frac{1}{c^2}\left(1-\frac{2MG}{c^2r}\right)^{-1}\mathrm{d}r^2 - \frac{r^2}{c^2}(\mathrm{d}\phi^2 + \sin^2\phi\mathrm{d}\theta^2) \tag{4.224b}$$

in which

- $\mathrm{d}\tau$ is the proper time (intrinsic time) unaffected by any gravitational field;
- $\mathrm{d}t$ is the apparent time (ordinary time) affected by a gravitational field in a rest frame of the observer. Generally, $\mathrm{d}t > \mathrm{d}\tau$ due to gravitational time dilation;
- $\mathrm{d}\Omega^2$ $\equiv \mathrm{d}\phi^2 + \sin^2\phi\ \mathrm{d}\theta^2$;
- r is the radius of spherical coordinates (r, ϕ, θ), as shown in Fig. 4.1;
- G is the universal gravitational constant (Newton's constant);
- M is the object mass;
- c is the light speed.

Two singularities of the Schwarzschild's solution in Eq. (4.224a) exist as the space distance ds increases infinitely. Besides $r = 0$ (an unavoidable case), the space metric coefficient must be

$$g_{rr} = \frac{1}{1-\frac{2MG}{c^2r}} \to \infty \tag{4.225}$$

Thus,

$$1 - \frac{2MG}{c^2 r} = 0 \tag{4.226}$$

The Schwarzschild's radius can be defined at the singularity by

$$r_S \equiv r\big|_{g_{rr}\to\infty} = \frac{2MG}{c^2} \tag{4.227}$$

The proper time dτ without effect of any gravitational field results at $r \geq r_S$, i.e., the clock runs slower by the factor given in Eq. (4.228) when it comes near the gravitation field (d$t >$ dτ). This effect is called the gravitational time dilation (Cahill 2013).

$$\mathrm{d}\tau < \sqrt{g_{00}}\mathrm{d}t = \sqrt{\left(1 - \frac{2MG}{c^2 r}\right)}\mathrm{d}t < \mathrm{d}t \tag{4.228}$$

According to Eq. (4.228), the ordinary time dt becomes infinite as r reaches the Schwarzschild's radius r_S. The clock would stop running near the black hole because dt goes to infinity.

$$\begin{aligned} \mathrm{d}t > \frac{1}{\sqrt{g_{00}}}\mathrm{d}\tau &= \frac{1}{\sqrt{\left(1 - \frac{2MG}{c^2 r}\right)}}\mathrm{d}\tau \\ &= \sqrt{g_{rr}}\mathrm{d}\tau \to \infty \text{ as } r \to r_S. \end{aligned} \tag{4.229}$$

4.11 Schwarzschild Black Hole

In the case of $r < r_S$, the uncharged spherically symmetric dwarf star (neutron star) with a mass M collapses within a cylinder of the radius r less than the Schwarzschild's radius r_S. This cylinder is called the Schwarzschild black hole, as shown in Fig. 4.11. According to Eq. (4.227), the Schwarzschild's radius r_S of the sun is about 2.95×10^3 m compared to its radius R of 6.95×10^8 m (approximately 235,000 times larger than r_S). Therefore, our Sun with the mass m $\sim R^3$ would produce a huge energy E = mc^2 before it disappears into the black hole in the very far future.

Using the Hamiltonian H consisting of the generalized momentum p_i, active variable $\dot{q}_i$, and the Lagrangian L, the total conserved energy of the photon trajectory in the gravitational field is calculated (Susskind 2012).

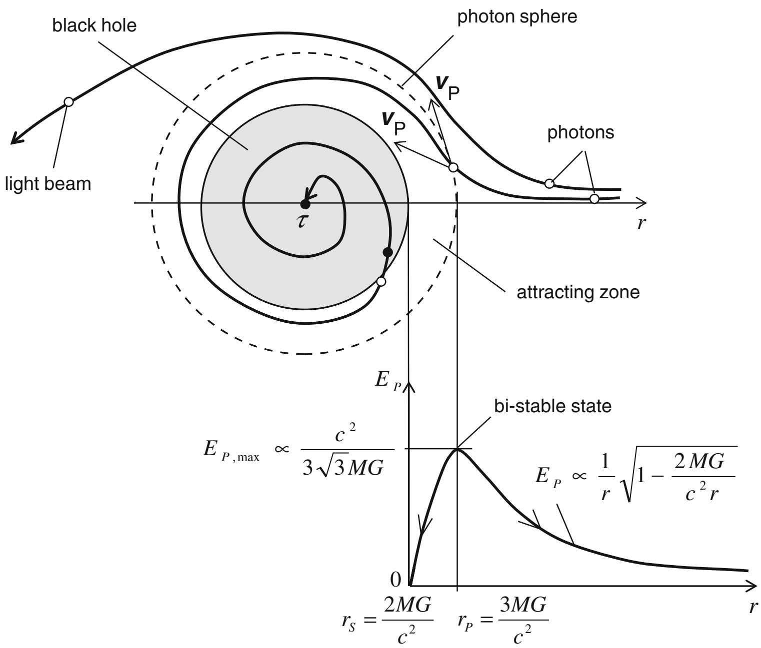

Fig. 4.11 Total energy of photons in the gravitational field of a curved space

$$\begin{aligned} H(q_i, p_i, t) &= \sum_i p_i \dot{q}_i - L = \sum_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) \dot{q}_i - L(q_i, \dot{q}_i, t) \\ &= T + V \\ &= \frac{p^2}{2m} + V(q) \\ &= E_P \propto \frac{1}{r}\sqrt{1 - \frac{2MG}{c^2 r}} \end{aligned} \tag{4.230}$$

where
the Lagrangian L is defined as

$$L \equiv T - V;$$

the generalized momentum and active variable can be written as

$$p_i = \frac{\partial L}{\partial \dot{q}_i};$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}.$$

In fact, the Hamiltonian is the total conserved energy of the kinetic and potential energy T and V of the system. Differentiating Eq. (4.230) with respect to r and calculating the second derivative at the extreme, the radius r_P results as the maximum energy of the photon trajectory occurs at the negative second derivative.

$$\begin{aligned} &\frac{\partial E_P}{\partial r} \propto \frac{\partial}{\partial r}\left(\frac{1}{r}\sqrt{1-\frac{2MG}{c^2 r}}\right) = 0 \\ &\Rightarrow r_P = \frac{3MG}{c^2} = \frac{3}{2} r_S \end{aligned} \tag{4.231}$$

The maximum energy can be calculated at $r = r_P$, as shown in Fig. 4.11.

$$E_{P,\max} \propto \frac{1}{r}\sqrt{1-\frac{2MG}{c^2 r}}\Bigg|_{r=r_P} \propto \frac{c^2}{3\sqrt{3}MG} \tag{4.232}$$

The photon with the maximum energy at the radius r_P locates in a bi-stable state. Through a small disturbance at this bi-stable state, either the photon trajectory is rejected outwards the photon sphere ($r > r_P$) or attracted toward the black hole ($r \to r_S$) in order to reach the stable state of minimum energy, as shown in Fig. 4.11. The sphere with the radius of $r_P = (3/2)\ r_S$ is called the photon sphere of the gravitational field. Furthermore, the photon energy equals zero at the Schwarzschild's radius r_S according to Eqs. (4.227 and 4.230).

Considering a photon moving from the outside ($r \gg r_P$) near to the photon sphere, the photon energy reaches a maximum at $r = r_P$. Between the photon sphere and the black hole ($r_S < r < r_P$), the photon energy reduces from the maximum to zero at the Schwarzschild's radius r_S of the black hole. If the photon trajectory (light beam) moves outside the photon sphere, the photon trajectory is accelerated with a bending radius around the black hole, as shown in Fig. 4.11. In another case, the photon trajectory enters the photon sphere with a velocity $\mathbf{v}_P$.

There are two possibilities for the light beam depending on the moving direction of $\mathbf{v}_P$ (Susskind 2012). Firstly, if the velocity $\mathbf{v}_P$ is tangent to the photon sphere surface, the light beam moves on the photon sphere surface in a stable condition. Secondly, if the radial component of $\mathbf{v}_P$ moves toward the black hole center (called the time line τ), the light beam becomes unstable and collapses itself into the black hole after a number of cycles because the balance between the energy created by the strong force and gravitational force fails. The energy of the photon trajectory reduces to zero as it moves toward the black hole ($r \to r_S$) under the very strong gravitational field of the black hole. This is to blame for the attraction of the photon trajectory to the black hole. In this case, the light beam has never came from the black hole back to the outside. The neutron star will be contracted in an infinitesimal point particle in the black hole under its huge gravitational field. Finally, the neutron star collapses in the black hole, crushing all its atoms into a highly dense ball of neutrons.

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Appendix A
Relations Between Covariant and Contravariant Bases

The contravariant basis vector $\mathbf{g}^k$ of the curvilinear coordinate of u^k at the point P is perpendicular to the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$, as shown in Fig. A.1. This contravariant basis $\mathbf{g}^k$ can be defined as

$$\alpha\, \mathbf{g}^k \equiv \mathbf{g}_i \times \mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^i} \times \frac{\partial \mathbf{r}}{\partial u^j} \tag{A.1}$$

where α is the scalar factor; $\mathbf{g}^k$ is the contravariant basis of the curvilinear coordinate of u^k.

Multiplying Eq. (A.1) by the covariant basis $\mathbf{g}_k$, the scalar factor α results in

$$\begin{aligned} &(\mathbf{g}_i \times \mathbf{g}_j).\, \mathbf{g}_k = \alpha(\mathbf{g}^k.\, \mathbf{g}_k) = \alpha\, \delta^k_k = \alpha \\ &\Rightarrow \alpha = (\mathbf{g}_i \times \mathbf{g}_j) \cdot \mathbf{g}_k \equiv \left[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k\right] \end{aligned} \tag{A.2}$$

The scalar triple product of the covariant bases can be written as

$$\alpha = [\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = (\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 = \sqrt{g} = J \tag{A.3}$$

where Jacobian J is the determinant of the covariant basis tensor $\mathbf{G}$.

The direction of the cross product vector in Eq. (A.1) is opposite if the dummy indices are interchanged with each other in Einstein summation convention. Therefore, the Levi-Civita permutation symbols (pseudo-tensor components) can be used in expression of the contravariant basis.

$$\begin{aligned} &\sqrt{g}\, \mathbf{g}^k = J\, \mathbf{g}^k = (\mathbf{g}_i \times \mathbf{g}_j) = -(\mathbf{g}_j \times \mathbf{g}_i) \\ &\Rightarrow \mathbf{g}^k = \frac{\varepsilon_{ijk}(\mathbf{g}_i \times \mathbf{g}_j)}{\sqrt{g}} = \frac{\varepsilon_{ijk}(\mathbf{g}_i \times \mathbf{g}_j)}{J} \end{aligned} \tag{A.4}$$

where the Levi-Civita permutation symbols are defined by

$$\begin{aligned} &\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation;} \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation;} \\ 0 & \text{if } i = j, \text{ or } i = k; \text{ or } j = k \end{cases} \\ &\Leftrightarrow \varepsilon_{ijk} = \frac{1}{2}(i-j) \cdot (j-k) \cdot (k-i) \quad \text{for } i,j,k = 1,2,3 \end{aligned} \tag{A.5}$$

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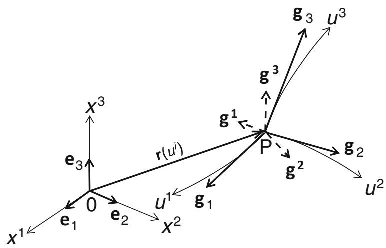

Fig. A.1 Covariant and contravariant bases of curvilinear coordinates

Thus, the cross product of the covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$ results from Eq. (A.4):

$$\begin{aligned}(\mathbf{g}_i \times \mathbf{g}_j) &= \varepsilon_{ijk}\sqrt{g}\,\mathbf{g}^k = \varepsilon_{ijk} J \mathbf{g}^k \equiv \hat{\varepsilon}_{ijk}\mathbf{g}^k \\ \Rightarrow \mathbf{g}^k &= \frac{\varepsilon_{ijk}(\mathbf{g}_i \times \mathbf{g}_j)}{\sqrt{g}} = \hat{\varepsilon}^{ijk}(\mathbf{g}_i \times \mathbf{g}_j) \\ \Rightarrow \hat{\varepsilon}_{ijk} &= (\mathbf{g}_i \times \mathbf{g}_j)\mathbf{g}_k\end{aligned} \tag{A.6}$$

The covariant permutation symbols in Eq. (A.6) can be defined as

$$\hat{\varepsilon}_{\mathrm{ijk}} = \begin{cases} +\sqrt{g} & \text{if } (i,j,k) \text{ is an even permutation;} \\ -\sqrt{g} & \text{if } (i,j,k) \text{ is an odd permutation;} \\ 0 & \text{if } i=j, \text{ or } i=k; \text{ or } j=k \end{cases} \tag{A.7}$$

The contravariant permutation symbols in Eq. (A.6) can be defined as

$$\hat{\varepsilon}^{ijk} = \begin{cases} +\frac{1}{\sqrt{g}} & \text{if } (i,j,k) \text{ is an even permutation;} \\ -\frac{1}{\sqrt{g}} & \text{if } (i,j,k) \text{ is an odd permutation;} \\ 0 & \text{if } i=j, \text{ or } i=k; \text{ or } j=k \end{cases} \tag{A.8}$$

The covariant basis vector $\mathbf{g}_k$ of the curvilinear coordinate of u^k at the point P is perpendicular to the contravariant bases $\mathbf{g}^i$ and $\mathbf{g}^j$, as shown in Fig. A.1. Therefore, the cross product of the contravariant bases $\mathbf{g}^i$ and $\mathbf{g}^j$ can be written as

$$\begin{aligned}(\mathbf{g}^i \times \mathbf{g}^j) &= \frac{\varepsilon_{ijk}}{\sqrt{g}}\mathbf{g}_k = \frac{\varepsilon_{ijk}}{J}\mathbf{g}_k \equiv \hat{\varepsilon}^{ijk}\mathbf{g}_k \\ \Rightarrow \hat{\varepsilon}^{ijk} &= (\mathbf{g}^i \times \mathbf{g}^j)\mathbf{g}^k\end{aligned} \tag{A.9}$$

Thus, the covariant basis results from Eq. (A.9):

$$\begin{aligned}\mathbf{g}_k &= \varepsilon_{ijk}\sqrt{g}(\mathbf{g}^i \times \mathbf{g}^j) = \varepsilon_{ijk} J(\mathbf{g}^i \times \mathbf{g}^j) \\ &= \hat{\varepsilon}_{ijk}(\mathbf{g}^i \times \mathbf{g}^j)\end{aligned} \tag{A.10}$$

Obviously, there are some relations between the covariant and contravariant permutation symbols:

$$\begin{aligned}\hat{\varepsilon}^{ijk}\hat{\varepsilon}_{ijk} &= 1 \ (\text{no summation}) \\ \hat{\varepsilon}_{ijk} &= \hat{\varepsilon}^{ijk} J^2 \ (\text{no summation})\end{aligned} \tag{A.11}$$

The tensor product of the covariant and contravariant permutation pseudo-tensors is a sixth-order tensor.

$$\hat{\varepsilon}^{ijk}\hat{\varepsilon}_{pqr} = \delta^{ijk}_{pqr} = \begin{cases} +1; & (i,j,k) \text{ and } (l,m,n) \text{ even permutation;} \\ -1; & (i,j,k) \text{ and } (l,m,n) \text{ odd permutation;} \\ 0; & \text{otherwise} \end{cases} \tag{A.12}$$

The sixth-order Kronecker tensor can be written in the determinant form:

$$\hat{\varepsilon}^{ijk}\hat{\varepsilon}_{pqr} = \delta^{ijk}_{pqr} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^k_p & \delta^k_q & \delta^k_r \end{vmatrix} \tag{A.13}$$

Using the tensor contraction rules with $k = r$, one obtains

$$\begin{aligned} \delta^{ij}_{pq} = \delta^{ijr}_{pqr} &= \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ \delta^r_p & \delta^r_q & \delta^r_r \end{vmatrix} = \begin{vmatrix} \delta^i_p & \delta^i_q & \delta^i_r \\ \delta^j_p & \delta^j_q & \delta^j_r \\ 0 & 0 & 1 \end{vmatrix} \\ \Rightarrow \hat{\varepsilon}^{ij}\hat{\varepsilon}_{pq} = \delta^{ij}_{pq} &= 1 \cdot \begin{vmatrix} \delta^i_p & \delta^i_q \\ \delta^j_p & \delta^j_q \end{vmatrix} = \delta^i_p\delta^j_q - \delta^i_q\delta^j_p \end{aligned} \tag{A.14}$$

Further contraction of Eq. (A.14) with $j = q$ gives

$$\begin{aligned} \hat{\varepsilon}^{iq}\hat{\varepsilon}_{pq} &= \delta^i_p\delta^q_q - \delta^i_q\delta^q_p \\ &= \delta^i_p\delta^q_q - \delta^i_p = 2\delta^i_p - \delta^i_p = \delta^i_p \quad \text{for } i,p = 1,2 \end{aligned} \tag{A.15}$$

From Eq. (A.15), the next contraction with $i = p$ gives

$$\begin{aligned} \hat{\varepsilon}^{pq}\hat{\varepsilon}_{pq} &= \delta^p_p \ (\text{summation over } p) \\ &= \delta^1_1 + \delta^2_2 = 2 \quad \text{for } p = 1,2 \end{aligned} \tag{A.16}$$

Similarly, contracting Eq. (A.13) with $k = r$; $j = q$, one has for a three-dimensional space.

$$\begin{aligned} \hat{\varepsilon}^{iq}\hat{\varepsilon}_{pq} &= \delta^i_p\delta^q_q - \delta^q_p\delta^i_q \\ &= \delta^i_p\delta^q_q - \delta^i_p = 3\delta^i_p - \delta^i_p = 2\delta^i_p \quad \text{for } i,p = 1,2,3 \end{aligned} \tag{A.17}$$

Contracting Eq. (A.17) with $i = p$, one obtains

$$\begin{aligned} \hat{\varepsilon}^{pq}\hat{\varepsilon}_{pq} &= 2\delta^p_p \ (\text{summation over } p) \\ &= 2(\delta^1_1 + \delta^2_2 + \delta^3_3) \\ &= 2(1 + 1 + 1) = 6 \quad \text{for } p = 1,2,3 \end{aligned} \tag{A.18}$$

The covariant metric tensor **M** can be written as

$$\mathbf{M} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{A.19}$$

where the covariant metric coefficients are defined by $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$.

The contravariant metric coefficients in the contravariant metric tensor $\mathbf{M}^{-1}$ result from inverting the covariant metric tensor **M**.

$$\mathbf{M}^{-1} = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} \tag{A.20}$$

where the contravariant metric coefficients are defined by $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$.

Thus, the relation between the covariant and contravariant metric coefficients can be written as

$$g^{ik} g_{kj} = g_{kj} g^{ik} = \delta^i_j \Leftrightarrow \mathbf{M}^{-1}\mathbf{M} = \mathbf{M}\mathbf{M}^{-1} = \mathbf{I} \tag{A.21}$$

In the case of $i \neq j$, all terms of $g^{ik} g_{kj}$ equal zero. Thus, only nine terms of $g^{ik} g_{ki}$ for $i = j$ remain in a three-dimensional space $\mathbf{R}^3$:

$$\begin{aligned} g^{ik} g_{ki} &= g^{1k} g_{k1} + g^{2k} g_{k2} + g^{3k} g_{k3} \quad \text{for } i, k = 1, 2, 3 \\ &= \delta^1_1 + \delta^2_2 + \delta^3_3 = \delta^i_i \quad \text{for } i = 1, 2, 3 \\ &= 1 + 1 + 1 = 3 \end{aligned} \tag{A.22}$$

The relation between the covariant and contravariant bases in the general curvilinear coordinates results in

$$\begin{aligned} & \mathbf{g}^{\mathrm{i}} . \mathbf{g}_{\mathrm{j}} = g^{ik} g_{kj} = \delta^i_j & \text{for } i \equiv j \\ \Rightarrow\ & \mathbf{g}^i . \mathbf{g}_i = \mathbf{g}^1 . \mathbf{g}_1 + \mathbf{g}^2 . \mathbf{g}_2 + \mathbf{g}^3 . \mathbf{g}_3 & \text{for } i = 1, 2, 3 \\ & = g^{ik} g_{ki} = \delta^i_i & \text{for } i, k = 1, 2, 3 \end{aligned} \tag{A.23}$$

According to Eq. (A.23), nine terms of $g^{ik} g_{ki}$ for k = 1, 2, 3 result in

$$\begin{cases} \mathbf{g}^1 \cdot \mathbf{g}_1 = g^{1k} g_{k1} = g^{11} g_{11} + g^{12} g_{21} + g^{13} g_{31} = \delta^1_1 & \text{for } i = 1; \\ \mathbf{g}^2 \cdot \mathbf{g}_2 = g^{2k} g_{k2} = g^{21} g_{12} + g^{22} g_{22} + g^{23} g_{32} = \delta^2_2 & \text{for } i = 2; \\ \mathbf{g}^3 \cdot \mathbf{g}_3 = g^{3k} g_{k3} = g^{31} g_{13} + g^{32} g_{23} + g^{33} g_{33} = \delta^3_3 & \text{for } i = 3. \end{cases} \tag{A.24}$$

The scalar product of the covariant and contravariant bases gives

$$\begin{aligned} \mathbf{g}^{(i)} \cdot \mathbf{g}_{(i)} &= \left|\mathbf{g}^{(i)}\right| \cdot \left|\mathbf{g}_{(i)}\right| \cos(\mathbf{g}^{(i)}, \mathbf{g}_{(i)}) \\ &= \sqrt{g^{(ii)}} \cdot \sqrt{g_{(ii)}} \cos(\mathbf{g}^{(i)}, \mathbf{g}_{(i)}) = 1 \end{aligned} \tag{A.25}$$

where the index (i) means no summation is carried out over i.

Equation (A.25) indicates that the product of the covariant and contravariant basis norms generally does not equal one in the curvilinear coordinates.

$$\sqrt{g^{(ii)}} \cdot \sqrt{g_{(ii)}} = \frac{1}{\cos(\mathbf{g}^{(i)}, \mathbf{g}_{(i)})} \geq 1 \tag{A.26}$$

In *orthogonal* coordinate systems, $\mathbf{g}^{(i)}$ is parallel to $\mathbf{g}_{(i)}$. Therefore, Eq. (A.26) becomes

$$\sqrt{g^{(ii)}} \cdot \sqrt{g_{(ii)}} = 1 \Rightarrow \sqrt{g^{(ii)}} = \frac{1}{\sqrt{g_{(ii)}}} = \frac{1}{h_i} \tag{A.27}$$

Appendix B
Physical Components of Tensors

The physical component of a tensor can be defined as the tensor component on its unitary covariant basis. Therefore, the covariant basis of the general curvilinear coordinates has to be normalized.

Dividing the covariant basis by its vector length, the unitary covariant basis (covariant-normalized basis) results in

$$\mathbf{g}_i^* = \frac{\mathbf{g}_i}{|\mathbf{g}_i|} = \frac{\mathbf{g}_i}{\sqrt{g_{(ii)}}} \Rightarrow |\mathbf{g}_i^*| = 1 \tag{B.1a}$$

The covariant basis norm $|\mathbf{g}_i|$ can be considered as a scale factor h_i without summation over (i).

$$h_i = |\mathbf{g}_i| = \sqrt{g_{(ii)}} \tag{B.1b}$$

Thus, the covariant basis can be related to its unitary covariant basis by the relation

$$\mathbf{g}_i = \sqrt{g_{(ii)}}\mathbf{g}_i^* = h_i\mathbf{g}_i^* \tag{B.2}$$

The contravariant basis can be related to its unitary covariant basis using Eqs. (2.47 and B.2).

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j = g^{ij}h_j\mathbf{g}_j^* \tag{B.3}$$

The contravariant second-order tensor can be written in the unitary covariant bases using Eq. (B.2).

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j = (T^{ij}h_ih_j)\,\mathbf{g}_i^*\mathbf{g}_j^* \equiv T^{*ij}\mathbf{g}_i^*\mathbf{g}_j^* \tag{B.4}$$

Thus, the physical contravariant tensor components denoted by *star* result in

$$T^{*ij} \equiv h_ih_jT^{ij} \tag{B.5}$$

The covariant second-order tensor can be written in the unitary contravariant bases using Eq. (B.3).

$$\mathbf{T} = T_{ij}\mathbf{g}^i\mathbf{g}^j = (T_{ij}g^{ik}g^{jl}h_kh_l)\,\mathbf{g}_k^*\mathbf{g}_l^* \equiv T_{ij}^*\mathbf{g}_k^*\mathbf{g}_l^* \tag{B.6}$$

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Similarly, the physical covariant tensor components denoted by *star* result in

$$T^*_{ij} \equiv g^{ik} g^{jl} h_k h_l T_{ij} \tag{B.7}$$

The mixed tensors can be written in the unitary covariant bases using Eqs. (B.2 and B.3)

$$\begin{aligned} \mathbf{T} &= T^i_j \mathbf{g}_i \mathbf{g}^j = T^i_j \mathbf{g}_i (g^{jk} \mathbf{g}_k) \\ &= T^i_j (h_i \mathbf{g}^*_i)(g^{jk} h_k \mathbf{g}^*_k) \\ &= (T^i_j g^{jk} h_i h_k) \mathbf{g}^*_i \mathbf{g}^*_k \\ &\equiv (T^i_j)^* \mathbf{g}^*_i \mathbf{g}^*_k \end{aligned} \tag{B.8}$$

Thus, the physical mixed tensor components denoted by *star* result in

$$(T^i_j)^* \equiv g^{jk} h_i h_k T^i_j \tag{B.9}$$

Analogously, the contravariant vector can be written using Eq. (B.2).

$$\begin{aligned} \mathbf{v} &= v^i \mathbf{g}_i = (v^i h_i) \mathbf{g}^*_i \\ &\equiv v^{*i} \mathbf{g}^*_i = \frac{v^{*i}}{h_i} \mathbf{g}_i \end{aligned} \tag{B.10}$$

Thus, the physical component of the contravariant vector **v** on the unitary basis $\mathbf{g}^*_i$ is defined as

$$v^{*i} \equiv h_i v^i = \sqrt{g_{(ii)}}\, v^i \tag{B.11}$$

The contravariant basis can be normalized dividing by its vector length without summation over (i).

$$\mathbf{g}^{*i} = \frac{\mathbf{g}^i}{|\mathbf{g}^i|} = \frac{\mathbf{g}^i}{\sqrt{g^{(ii)}}} \tag{B.12}$$

where $g^{(ii)}$ is the contravariant metric coefficient that results from Eq. (A.20).

Using Eq. (B.12), the covariant vector **v** can be written as

$$\mathbf{v} = v_i \mathbf{g}^i = v_i \sqrt{g^{(ii)}}\, \mathbf{g}^{*i} \equiv v^*_i \mathbf{g}^{*i} \tag{B.13}$$

Thus, the physical component of the covariant vector **v** results in

$$v^*_i = v_i \sqrt{g^{(ii)}} \tag{B.14}$$

According to Eq. (A.27), Eq. (B.14) can be rewritten in *orthogonal* coordinate systems:

$$v^*_i = \sqrt{g^{(ii)}}\, v_i = \frac{1}{\sqrt{g_{(ii)}}} v_i = \frac{1}{h_i} v_i \tag{B.15}$$

Using Eq. (B.3), the covariant vector can be written in

$$\begin{aligned}\mathbf{v} &= v_i\mathbf{g}^i = v_i g^{ij}\mathbf{g}_j \\ &= (v_i g^{ij} h_j)\mathbf{g}_j^* \equiv v_j^* \mathbf{g}_j^* \\ &= \frac{v_j^*}{g^{ij}h_j}\mathbf{g}^i\end{aligned} \tag{B.16}$$

Thus, the physical contravariant vector component of **v** on the unitary basis $\mathbf{g}_j^*$ can be defined as

$$v_j^* \equiv g^{ij} h_j v_i \tag{B.17}$$

Furthermore, the vector **v** can be written in both covariant and contravariant bases.

$$\begin{aligned}&\mathbf{v} = v_j\mathbf{g}^j = v^i\mathbf{g}_i \\ &\Rightarrow (v_j\mathbf{g}^j).\mathbf{g}_k = (v^i\mathbf{g}_i).\mathbf{g}_k \\ &\Rightarrow v_j\delta_k^j = v^i g_{ik} \\ &\Rightarrow v_k = v^i g_{ik}\end{aligned} \tag{B.18}$$

Interchanging i with j and k with i, one obtains

$$v_i = v^j g_{ij} \tag{B.19}$$

Only in *orthogonal* coordinate systems, we have

$$g_{ij} = 0 \quad \text{for } i \neq j;\ g_{(ii)} = h_i^2 \tag{B.20}$$

Thus, one obtains from Eq. (B.19)

$$\begin{aligned}v_i &= v^j g_{ij} = v^1 g_{i1} + v^2 g_{i2} + \cdots + v^N g_{iN} \\ &= v^i g_{(ii)} = v^i h_i^2\end{aligned} \tag{B.21}$$

Substituting Eq. (B.11) into Eq. (B.21), one obtains Eq. (B.22) that is equivalent to Eq. (B.15).

$$v_i = v^i h_i^2 = \left(\frac{v^{*i}}{h_i}\right) h_i^2 = h_i v^{*i} \tag{B.22}$$

Appendix C
Nabla Operators

Some useful Nabla operators are listed in Cartesian and general curvilinear coordinates:

1. **Gradient of an invariant** $\boldsymbol{f}$

   - Cartesian coordinate $\{x^i\}$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{e}_x + \frac{\partial f}{\partial y}\mathbf{e}_y + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{C.1}$$

   - General curvilinear coordinate $\{u^i\}$

$$\nabla f = f_{,i}\mathbf{g}^i = \frac{\partial f}{\partial u^i}\mathbf{g}^i = \frac{\partial f}{\partial u^i} g^{ij}\mathbf{g}_j \tag{C.2}$$

2. **Gradient of a vector v**

   - General curvilinear coordinate $\{u^i\}$

$$\begin{aligned} \nabla \mathbf{v} &= \left(v^k_{,i} + v^j \Gamma^k_{ij} \right) \mathbf{g}_k \mathbf{g}^i = v^k\big|_i \mathbf{g}_k \mathbf{g}^i \\ &= \left(v_{k,i} - v_j \Gamma^j_{ik} \right) \mathbf{g}^k \mathbf{g}^i = v_k\big|_i \mathbf{g}^k \mathbf{g}^i \end{aligned} \tag{C.3}$$

3. **Divergence of a vector v**

   - Cartesian coordinate $\{x^i\}$

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{C.4}$$

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- General curvilinear coordinate $\{u^i\}$

$$\begin{aligned}\nabla.\mathbf{v} &= v^i|_i \equiv (v^i_{,i} + v^j\Gamma^i_{ij}) \\ &= \frac{1}{J}\frac{\partial(Jv^i)}{\partial u^i} = J^{-1}(Jv^i)_{,i} \\ \nabla\cdot\mathbf{v} &= v_k|_i g^{ki} = (v_{k,i} - v_j\Gamma^j_{ik})\mathbf{g}^k\cdot\mathbf{g}^i\end{aligned} \tag{C.5}$$

4. **Gradient of a second-order tensor T**

- General curvilinear coordinate $\{u^i\}$ for a covariant second-order tensor

$$\begin{aligned}\nabla\mathbf{T} &= (T_{ij,k} - \Gamma^m_{ik}T_{mj} - \Gamma^m_{jk}T_{im})\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k \\ &= T_{ij}|_k\mathbf{g}^i\mathbf{g}^j\mathbf{g}^k\end{aligned} \tag{C.6}$$

- General curvilinear coordinate $\{u^i\}$ for a contravariant second-order tensor

$$\begin{aligned}\nabla\mathbf{T} &= (T^{ij}_{,k} + \Gamma^i_{km}T^{mj} + \Gamma^j_{km}T^{im})\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k \\ &= T^{\mathrm{ij}}|_k\mathbf{g}_i\mathbf{g}_j\mathbf{g}^k\end{aligned} \tag{C.7}$$

- General curvilinear coordinate $\{u^i\}$ for a mixed second-order tensor

$$\begin{aligned}\nabla\mathbf{T} &= (T^i_{j,k} + \Gamma^i_{km}T^m_j - \Gamma^m_{jk}T^i_m)\mathbf{g}_i\mathbf{g}^j\mathbf{g}^k \\ &= T^i_j|_k\mathbf{g}_i\mathbf{g}^j\mathbf{g}^k\end{aligned} \tag{C.8}$$

5. **Divergence of a second-order tensor T**

- General curvilinear coordinate $\{u^i\}$ for a covariant second-order tensor

$$\begin{aligned}\nabla.\mathbf{T} &= (T_{ij,k} - \Gamma^m_{ik}T_{mj} - \Gamma^m_{jk}T_{im})\mathbf{g}^i(\mathbf{g}^j.\mathbf{g}^k) \\ &\equiv T_{ij}|_k g^{jk}\mathbf{g}^i\end{aligned} \tag{C.9}$$

- General curvilinear coordinate $\{u^i\}$ for a contravariant second-order tensor

$$\begin{aligned}\nabla\cdot\mathbf{T} &= (T^{ij}_{,k} + \Gamma^i_{km}T^{mj} + \Gamma^j_{km}T^{im})\delta^k_i\mathbf{g}_j \\ &= (T^{ij}_{,i} + \Gamma^i_{im}T^{mj} + \Gamma^j_{im}T^{im})\mathbf{g}_j \\ &\equiv T^{ij}|_i\mathbf{g}_j\end{aligned} \tag{C.10}$$

- General curvilinear coordinate $\{u^i\}$ for a mixed second-order tensor

$$\begin{aligned}\nabla\cdot\mathbf{T} &= (T^i_{j,k} + \Gamma^i_{km}T^m_j - \Gamma^m_{jk}T^i_m)\delta^k_i\mathbf{g}^j \\ &= (T^i_{j,i} + \Gamma^i_{im}T^m_j - \Gamma^m_{ji}T^i_m)\mathbf{g}^j \\ &\equiv T^i_j|_i\mathbf{g}^j = T^i_j|_i g^{jk}\mathbf{g}_k\end{aligned} \tag{C.11}$$

6. **Curl of a vector v**

- Cartesian coordinate $\{x^i\}$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \tag{C.12}$$

The curl of $\mathbf{v}$ results from calculating the determinant of Eq. (C.12).

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathbf{e}_z \tag{C.13}$$

- General curvilinear coordinate $\{u^i\}$

$$\nabla \times \mathbf{v} = \hat{\varepsilon}^{ijk} v_{j,i} \mathbf{g}_k \tag{C.14}$$

The contravariant permutation symbol is defined by

$$\hat{\varepsilon}^{ijk} = \begin{cases} +J^{-1} & \text{if } (i,j,k) \text{ is an even permutation;} \\ -J^{-1} & \text{if } (i,j,k) \text{ is an odd permutation;} \\ 0 & \text{if } i = j, \text{ or } i = k; \text{ or } j = k \end{cases} \tag{C.15}$$

where J is the Jacobian.

7. **Laplacian of an invariant *f***

- Cartesian coordinate $\{x^i\}$

$$\nabla^2 f \equiv \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{C.16}$$

- General curvilinear coordinate $\{u^i\}$

$$\begin{aligned} \nabla^2 f \equiv \Delta f &= (f_{,ij} - f_{,k}\Gamma_{ij}^k) g^{ij} \\ &= (v_{i,j} - v_k \Gamma_{ij}^k) g^{ij} \equiv v_i\big|_j g^{ij} \end{aligned} \tag{C.17}$$

where the covariant vector component and its covariant derivative with respect to u^k are defined by

$$v_i = f_{,i} = \frac{\partial f}{\partial u^i}; \quad v_k = f_{,k} = \frac{\partial f}{\partial u^k}; \quad v_{i,j} = f_{,ij} = \frac{\partial^2 f}{\partial u^i \partial u^j} \tag{C.18}$$

8. **Calculation rules of the Nabla operators**

$$\text{Div Grad} f = \nabla \cdot (\nabla f) = \nabla^2 f = \Delta f (\text{Laplacian}) \tag{C.19}$$

$$\text{Curl Grad} f = \nabla \times (\nabla f) = 0 \tag{C.20}$$

$$\text{Div Curl } \mathbf{v} = \nabla \cdot (\nabla \times \mathbf{v}) = 0 \tag{C.21}$$

$$\Delta(fg) = f \Delta g + 2\nabla f \cdot \nabla g + g \Delta f \tag{C.22}$$

$$\text{Curl Curl } \mathbf{v} = \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \Delta \mathbf{v} (\text{Curl identity}) \tag{C.23}$$

The Laplacian of **v** in Eq. (C.23) is computed in the tensor formulation for general curvilinear coordinates.

$$\begin{aligned} \text{Div Grad } \mathbf{v} &= \text{Laplacian } \mathbf{v} = \Delta \mathbf{v} \\ &= \nabla \cdot (\nabla \mathbf{v}) = \nabla^2 \mathbf{v} \\ &= \left(v^i \big|_{j,k} - v^i \big|_p \Gamma^p_{jk} + v^p \big|_j \Gamma^i_{pk} \right) g^{jk} \mathbf{g}_i \\ &\equiv v^i \big|_{jk} g^{jk} \mathbf{g}_i \end{aligned} \tag{C.24}$$

Appendix D
Essential Tensors

Derivative of the covariant basis

$$\mathbf{g}_{i,j} = \Gamma_{ij}^{k}\mathbf{g}_{k} \tag{D.1}$$

Derivative of the contravariant basis

$$\mathbf{g}_{,j}^{i} = \frac{\partial \mathbf{g}^{i}}{\partial u^{j}} \equiv \hat{\Gamma}_{jk}^{i}\mathbf{g}^{k} = -\Gamma_{jk}^{i}\mathbf{g}^{k} \tag{D.2}$$

Derivative of the covariant metric coefficient

$$g_{ij,k} = \Gamma_{ik}^{p} g_{pj} + \Gamma_{jk}^{p} g_{p\,i} \tag{D.3}$$

First-kind Christoffel symbol

$$\begin{aligned} \Gamma_{ijk} &= \frac{1}{2}(g_{ik,j} + g_{jk,i} - g_{ij,k}) = g_{lk}\Gamma_{ij}^{l} \\ \Rightarrow \Gamma_{ij}^{l} &= g^{lk}\Gamma_{ijk} \end{aligned} \tag{D.4}$$

Second-kind Christoffel symbol based on the covariant basis

$$\Gamma_{ij}^{k} = \mathbf{g}_{i,j} \cdot \mathbf{g}^{k} = g^{kl}\Gamma_{ijl} \tag{D.5}$$

$$\Gamma_{ij}^{k} = \frac{\partial u^{k}}{\partial x^{p}} \cdot \frac{\partial^{2} x^{p}}{\partial u^{i} \partial u^{j}} = \Gamma_{ji}^{k} \tag{D.6}$$

$$\Gamma_{ij}^{k} = g^{kp}\frac{1}{2}(g_{ip,j} + g_{jp,i} - g_{ij,p}) \tag{D.7}$$

$$\Gamma_{ij}^{i} = \frac{1}{J}\frac{\partial J}{\partial u^{j}} = \frac{\partial(\ln J)}{\partial u^{j}} \tag{D.8}$$

Second-kind Christoffel symbol based on the contravariant basis

$$\hat{\Gamma}_{kj}^{i} = -\Gamma_{kj}^{i} = \hat{\Gamma}_{jk}^{i} \tag{D.9}$$

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Covariant derivative of covariant first-order tensors

$$T_i\big|_j = T_{i,j} - \Gamma^k_{ij} T_k = \mathbf{T}_{,j} \cdot \mathbf{g}_i \tag{D.10}$$

Covariant derivative of contravariant first-order tensors

$$T^i\big|_j = T^i_{,j} + \Gamma^i_{jk} T^k = \mathbf{T}_{,j} \cdot \mathbf{g}^i \tag{D.11}$$

Covariant derivative of covariant and contravariant second-order tensors

$$\begin{aligned} T_{ij}|_k &= T_{ij,k} - \Gamma^m_{ik} T_{mj} - \Gamma^m_{jk} T_{im} \\ T^{ij}|_k &= T^{ij}_{,k} + \Gamma^i_{km} T^{mj} + \Gamma^j_{km} T^{im} \end{aligned} \tag{D.12}$$

Covariant derivative of mixed second-order tensors

$$\begin{aligned} T^i_j|_k &= T^i_{j,k} + \Gamma^i_{km} T^m_j - \Gamma^m_{jk} T^i_m \\ T^j_i|_k &= T^j_{i,k} + \Gamma^j_{km} T^m_i - \Gamma^m_{ik} T^j_m \end{aligned} \tag{D.13}$$

Second covariant derivative of covariant first-order tensors

$$\begin{aligned} T_i\big|_{kj} = T_{i,jk} &- \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} T_{m,j} \\ &- \Gamma^m_{ij} T_{m,k} + \Gamma^m_{ij} \Gamma^n_{mk} T_n \\ &- \Gamma^m_{jk} T_{i,m} + \Gamma^m_{jk} \Gamma^n_{im} T_n \end{aligned} \tag{D.14}$$

Second covariant derivative of the contravariant vector

$$v^k|_{lm} \equiv v^k\big|_{l,m} - v^k\big|_p \Gamma^p_{lm} + v^p|_l \Gamma^k_{pm} \tag{D.15}$$

where

$$v^k\big|_{l,m} = \left(v^k|_l\right)_{,m} \equiv v^k_{,lm} + v^n_{,m} \Gamma^k_{nl} + v^n \Gamma^k_{nl,m} \tag{D.16}$$

$$v^k\big|_p \equiv v^k_{,p} + v^n \Gamma^k_{np} \tag{D.17}$$

$$v^p|_l \equiv v^p_{,l} + v^n \Gamma^p_{nl} \tag{D.18}$$

Riemann–Christoffel tensor

$$R^n_{ijk} \equiv \Gamma^n_{ik,j} - \Gamma^n_{ij,k} + \Gamma^m_{ik} \Gamma^n_{mj} - \Gamma^m_{ij} \Gamma^n_{mk} \tag{D.19}$$

Riemann curvature tensor

$$R_{lijk} \equiv g_{\ln} R^n_{ijk} \tag{D.20}$$

First-kind Ricci tensor

$$\begin{aligned} R_{ij} &= \frac{\partial \Gamma^k_{ik}}{\partial u^j} - \frac{\partial \Gamma^k_{ij}}{\partial u^k} - \Gamma^r_{ij}\Gamma^k_{rk} + \Gamma^r_{ik}\Gamma^k_{rj} \\ &= \frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \frac{1}{J}\frac{\partial (J\Gamma^k_{ij})}{\partial u^k} + \Gamma^r_{ik}\Gamma^k_{rj} \end{aligned} \tag{D.21}$$

Second-kind Ricci tensor

$$\begin{aligned} R^i_j &\equiv g^{ik} R_{kj} \\ &= g^{ik}\left(\frac{\partial^2(\ln J)}{\partial u^k \partial u^j} - \frac{1}{J}\frac{\partial (J\Gamma^m_{kj})}{\partial u^m} + \Gamma^m_{kn}\Gamma^n_{mj}\right) \end{aligned} \tag{D.22}$$

Ricci curvature

$$R = g^{ij}\left(\frac{\partial^2(\ln J)}{\partial u^i \partial u^j} - \frac{1}{J}\frac{\partial (J\Gamma^k_{ij})}{\partial u^k} + \Gamma^k_{ir}\Gamma^r_{kj}\right) \tag{D.23}$$

Einstein tensor

$$\begin{aligned} G^i_j &\equiv R^i_j - \tfrac{1}{2}\delta^i_j R = g^{ik} G_{kj} \\ G_{ij} &= g_{ik} G^k_j = R_{ij} - \tfrac{1}{2} g_{ij} R \end{aligned} \tag{D.24}$$

$$G^i_j\Big|_i = 0 \tag{D.25}$$

First fundamental form

$$\begin{aligned} I &= \mathrm{E}\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2; \\ \mathbf{M} = (g_{ij}) &\equiv \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{r}_u\mathbf{r}_u & \mathbf{r}_u\mathbf{r}_v \\ \mathbf{r}_v\mathbf{r}_u & \mathbf{r}_v\mathbf{r}_v \end{bmatrix} \end{aligned} \tag{D.26}$$

Second fundamental form

$$\begin{aligned} II &= L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2; \\ \mathbf{H} = (h_{ij}) &\equiv \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{uu}\mathbf{n} & \mathbf{r}_{uv}\mathbf{n} \\ \mathbf{r}_{uv}\mathbf{n} & \mathbf{r}_{vv}\mathbf{n} \end{bmatrix} \end{aligned} \tag{D.27}$$

Gaussian curvature of a curvilinear surface

$$K = \kappa_1\kappa_2 = \frac{LN - M^2}{EG - F^2} = \frac{\det(h_{ij})}{\det(g_{ij})} \tag{D.28}$$

Mean curvature of a curvilinear surface

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2) = \frac{EN - 2MF + LG}{2(EG - F^2)} \tag{D.29}$$

Unit normal vector of a curvilinear surface

$$\mathbf{n} = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{|\mathbf{g}_1 \times \mathbf{g}_2|} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{\det(g_{ij})}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{EG - F^2}} \tag{D.30}$$

Differential of a surface area

$$\begin{aligned} \mathrm{d}A &= |\mathbf{g}_1 \times \mathbf{g}_2| \mathrm{d}u\mathrm{d}v = \sqrt{g_{11}g_{22} - (g_{12})^2}\mathrm{d}u\mathrm{d}v \\ &= \sqrt{\det(g_{ij})}\mathrm{d}u\mathrm{d}v = \sqrt{EG - F^2}\mathrm{d}u\mathrm{d}v \end{aligned} \tag{D.31}$$

Gauss derivative equations

$$\begin{aligned} &\mathbf{g}_{i,j} = \Gamma_{ij}^k \mathbf{g}_k + h_{ij}\mathbf{g}_3 = \Gamma_{ij}^k \mathbf{g}_k + h_{ij}\mathbf{n} \\ &\Leftrightarrow \mathbf{g}_i\big|_j \equiv \mathbf{g}_{i,j} - \Gamma_{ij}^k \mathbf{g}_k = h_{ij}\mathbf{n} \end{aligned} \tag{D.32}$$

Weingarten's equations

$$\mathbf{n}_i = -h_i^j \mathbf{g}_j = -(h_{ik}g^{jk})\mathbf{g}_j \tag{D.33}$$

Codazzi's equations

$$K_{ij,k} = K_{ik,j} \Rightarrow \left(K_{11,2} = K_{12,1}\,;\; K_{21,2} = K_{22,1}\right) \tag{D.34}$$

Gauss equations

$$\begin{aligned} K &= \det\left(K_i^j\right) = \left(K_1^1 K_2^2 - K_2^1 K_1^2\right) \\ &= \frac{K_{11}K_{22} - K_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{R_{1212}}{g} \end{aligned} \tag{D.35}$$

Appendix E
Euclidean and Riemannian Manifolds

In the following appendix, we summarize fundamental notations and basic results from vector analysis in Euclidean and Riemannian manifolds. This section can be written informally and is intended to remind the reader of some fundamentals of vector analysis in general curvilinear coordinates. For the sake of simplicity, we abstain from being mathematically rigorous. Therefore, we recommend some literature given in References for the mathematically interested reader.

E.1 N-dimensional Euclidean Manifold

N-dimensional Euclidean manifold $\mathbf{E}^N$ can be represented by two kinds of coordinate systems: Cartesian (orthonormal) and curvilinear (non-orthogonal) coordinate systems with N dimensions. Lines, curves, and surfaces can be considered as subsets of Euclidean manifold. Two lines or two curves can generate a flat (planes) and curvilinear surface (cylindrical and spherical surfaces), respectively. Both kinds of surfaces can be embedded in Euclidean space.

E.1.1 Vector in Cartesian Coordinates

Cartesian coordinates are an orthonormal coordinate system in which the bases (**i**, **j**, **k**) are mutually perpendicular (*orthogonal*) and unitary (*normalized* vector length). The orthonormal bases (**i**, **j**, **k**) are fixed in Cartesian coordinates. Any vector could be described by its components and the relating bases in Cartesian coordinates.

The vector **r** can be written in Euclidean space $\mathbf{E}^3$ (three-dimensional space) in Cartesian coordinates (cf. Fig. E.1).

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{E.1}$$

H. Nguyen-Schäfer and J.-P. Schmidt, *Tensor Analysis and Elementary Differential Geometry for Physicists and Engineers*, Mathematical Engineering 21,
DOI: 10.1007/978-3-662-43444-4,

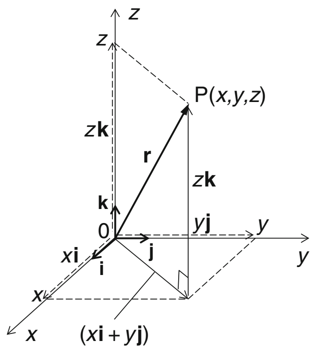

Fig. E.1 Vector **r** in Cartesian coordinates

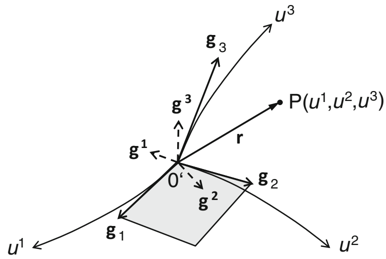

Fig. E.2 Bases of the curvilinear coordinates

where

x, y, z	are the vector components in the coordinate system (x, y, z);
i, **j**, **k**	are the orthonormal bases of the corresponding coordinates.

The vector length of **r** can be computed using the Pythagorean theorem as

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} \geq 0 \tag{E.2}$$

E.1.2 Vector in Curvilinear Coordinates

We consider a curvilinear coordinate system (u^1, u^2, u^3) of Euclidean space $\mathbf{E}^3$, i.e., a coordinate system which is generally non-orthogonal and non-unitary (non-orthonormal basis). By abuse of notation, we denote the basis vector simply basis.

In other words, the bases are not mutually perpendicular and their vector lengths are not equal to one (Klingbeil 1966; Nayak 2012). In the curvilinear coordinate system (u^1, u^2, u^3), there are three covariant bases $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{g}_3$ and three contravariant bases $\mathbf{g}^1$, $\mathbf{g}^2$, and $\mathbf{g}^3$ at the origin $0'$, as shown in Fig. E.2. Generally, the origin $0'$ of the curvilinear coordinates could move everywhere in Euclidean space; therefore, the bases of the curvilinear coordinates only depend on each considered origin 0′. For this reason, the bases are not fixed in the whole curvilinear coordinates such as in Cartesian coordinates, as displayed in Fig. E.1.

The vector **r** of the point $P(u^1, u^2, u^3)$ can be written in the covariant and contravariant bases:

$$\begin{aligned} \mathbf{r} &= u^1\mathbf{g}_1 + u^2\mathbf{g}_2 + u^3\mathbf{g}_3 \\ &= u_1\mathbf{g}^1 + u_2\mathbf{g}^2 + u_3\mathbf{g}^3 \end{aligned} \tag{E.3}$$

where

u^1, u^2, u^3 are the vector contravariant components of the coordinates (u^1, u^2, u^3);

$\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ are the covariant bases of the coordinate system (u^1, u^2, u^3);

u_1, u_2, u_3 are the vector covariant components of the coordinates (u^1, u^2, u^3);

$\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$ are the contravariant bases of the coordinate system (u^1, u^2, u^3).

The covariant basis $\mathbf{g}_i$ can be defined as the tangential vector to the corresponding curvilinear coordinate u^i for $i = 1, 2, 3$. Both bases $\mathbf{g}_1$ and $\mathbf{g}_2$ generate a tangential surface to the curvilinear surface (u^1u^2) at the considered origin 0′, as shown in Fig. E.2. Note that the basis $\mathbf{g}_1$ is not perpendicular to the bases $\mathbf{g}_2$ and $\mathbf{g}_3$. However, the contravariant basis $\mathbf{g}^3$ is perpendicular to the tangential surface $(\mathbf{g}_1\mathbf{g}_2)$ at the origin 0′. Generally, the contravariant basis $\mathbf{g}^k$ results from the cross product of the other covariant bases $(\mathbf{g}_i \times \mathbf{g}_j)$.

$$\alpha\,\mathbf{g}^k = \mathbf{g}_i \times \mathbf{g}_j \quad \text{for } i, j, k = 1, 2, 3 \tag{E.4a}$$

where α is a scalar factor (scalar triple product) given in Eq. (1.43).

$$\begin{aligned} \alpha &= (\mathbf{a}_i \times \mathbf{a}_j) \cdot \mathbf{a}_k \\ &\equiv \left[\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k\right] \end{aligned} \tag{E.4b}$$

Thus,

$$\mathbf{g}^1 = \frac{\mathbf{g}_2 \times \mathbf{g}_3}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]};\ \mathbf{g}^2 = \frac{\mathbf{g}_3 \times \mathbf{g}_1}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]};\ \mathbf{g}^3 = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \tag{E.4c}$$

E.1.3 Orthogonal and Orthonormal Coordinates

The coordinate system is called orthogonal if its bases are mutually perpendicular, as displayed in Fig. E.1. The dot product of two orthonormal bases is defined as

$$\begin{aligned}\mathbf{i}\,.\,\mathbf{j} &= |\mathbf{i}|\cdot|\mathbf{j}|\cdot\cos(\mathbf{i},\mathbf{j})\\ &= (1)\cdot(1)\cdot\cos\left(\frac{\pi}{2}\right)\\ &= 0\end{aligned}\tag{E.5}$$

Thus,

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0\tag{E.6}$$

If the length of each basis equals 1, the bases are unitary vectors.

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1\tag{E.7}$$

If the coordinate system satisfies both conditions (E.6) and (E.7), it is called the orthonormal coordinate system, which exists in Cartesian coordinates.

Therefore, the vector length in the orthonormal coordinate system results from

$$\begin{aligned}|\mathbf{r}|^2 &= \mathbf{r}\cdot\mathbf{r}\\ &= (x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k})\cdot(x\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\\ &= x^2(\mathbf{i}\cdot\mathbf{i}) + xy(\mathbf{i}\cdot\mathbf{j}) + xz(\mathbf{i}\cdot\mathbf{k})\\ &\quad + yx(\mathbf{j}\cdot\mathbf{i}) + y^2(\mathbf{j}\cdot\mathbf{j}) + yz(\mathbf{j}\cdot\mathbf{k})\\ &\quad + zx(\mathbf{k}\cdot\mathbf{i}) + zy(\mathbf{k}\cdot\mathbf{j}) + z^2(\mathbf{k}\cdot\mathbf{k})\end{aligned}\tag{E.8}$$

Due to Eqs. (E.6 and E.7), the vector length in Eq. (E.8) becomes

$$|\mathbf{r}|^2 = x^2 + y^2 + z^2 \Rightarrow |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}\tag{E.9}$$

The cross product (called vector product) of a pair of bases of the orthonormal coordinate system is (informally) given by means of *right-handed rule*; that is, if the right-hand fingers move in the rotating direction from the basis **j** to the basis **k**, the thumb will point in the direction of the basis $\mathbf{i} = \mathbf{j}\times\mathbf{k}$. The bases (**i**, **j**, **k**) form a right-handed triple.

$$\begin{cases}\mathbf{i} = \mathbf{j}\times\mathbf{k} = -\mathbf{k}\times\mathbf{j}\\ \mathbf{j} = \mathbf{k}\times\mathbf{i} = -\mathbf{i}\times\mathbf{k}\\ \mathbf{k} = \mathbf{i}\times\mathbf{j} = -\mathbf{j}\times\mathbf{i}\end{cases}\tag{E.10}$$

The cross product of two orthonormal bases can be defined as

$$\begin{aligned}|\mathbf{i}\times\mathbf{j}| &= |\mathbf{i}|\cdot|\mathbf{j}|\cdot\sin(\mathbf{i},\mathbf{j})\\ &= (1)\cdot(1)\cdot\sin\left(\frac{\pi}{2}\right)\\ &= |\mathbf{k}|\end{aligned}\tag{E.11}$$

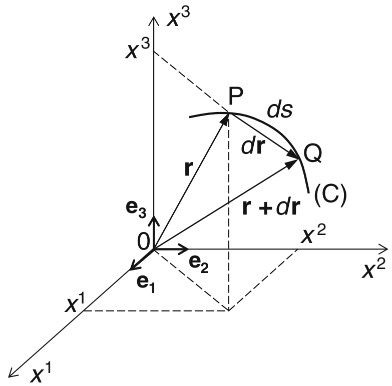

Fig. E.3 Arc length ds of *P* and *Q* in Cartesian coordinates

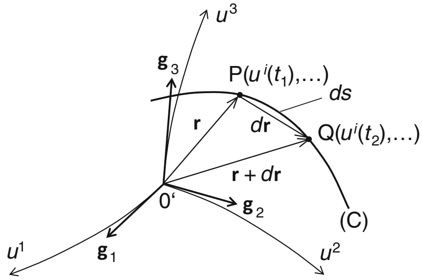

Fig. E.4 Arc length ds of *P* and *Q* in the curvilinear coordinates

E.1.4 Arc Length Between Two Points in a Euclidean Manifold

We consider two points $P(x^1, x^2, x^3)$ and $Q(x^1, x^2, x^3)$ in Euclidean space $\mathbf{E}^3$ in Cartesian and curvilinear coordinate systems, as shown in Figs. E.3 and E.4. Both points P and Q have three components x^1, x^2, and x^3 in Cartesian coordinates ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$). To simplify some mathematically written expressions, the coordinates x, y, and z in Cartesian coordinates can be transformed into x^1, x^2, and x^3; the bases ($\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$) turn to ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$).

We now turn to the notation of the differential d$\mathbf{r}$ of a vector $\mathbf{r}$. The differential d$\mathbf{r}$ can be expressed using the Einstein summation convention (Klingbeil 1966; Kay 2011):

$$\begin{aligned} \mathrm{d}\mathbf{r} &\equiv \mathbf{e}_\mathrm{i}\mathrm{d}x^i \quad \text{for } i = 1, 2, 3 \\ &= \sum_{i=1}^{3} \mathbf{e}_i \mathrm{d}x^i \end{aligned} \tag{E.12}$$

The Einstein summation convention used in Eq. (E.12) indicates that d$\mathbf{r}$ equals the sum of $\mathbf{e}_i\, \mathrm{d}x^i$ by running the dummy index i from 1 to 3.

The *arc length* ds between the points P and Q (cf. Fig. E.3) can be calculated by the dot product of two differentials.

$$\begin{aligned}(\mathrm{d}s)^2 &= \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r}\\ &= (\mathbf{e}_i\mathrm{d}x^i)\cdot(\mathbf{e}_j\mathrm{d}x^j)\\ &= (\mathbf{e}_i\cdot\mathbf{e}_j)\mathrm{d}x^i\cdot\mathrm{d}x^j\\ &= \mathrm{d}x^i\cdot\mathrm{d}x^i \quad \text{for } i=1,2,3.\end{aligned} \tag{E.13}$$

Thus, the arc length in the orthonormal coordinate system results in

$$\begin{aligned}\mathrm{d}s &= \sqrt{\mathrm{d}x^i\cdot\mathrm{d}x^i} \quad \text{for } i=1,2,3.\\ &= \sqrt{(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2}\end{aligned} \tag{E.14}$$

The points P and Q have three components u^1, u^2, and u^3 in the curvilinear coordinate system with the basis ($\mathbf{g_1}$, $\mathbf{g_2}$, $\mathbf{g_3}$) in Euclidean 3-space, as displayed in Fig. E.4. The location vector $\mathbf{r}(u^1,u^2,u^3)$ of the point P is a function of u^i. Therefore, the differential d**r** of the vector **r** can be rewritten in a linear formulation of du^i.

$$\begin{aligned}\mathrm{d}\mathbf{r} &= \frac{\partial\mathbf{r}}{\partial u^i}\mathrm{d}u^i\\ &\equiv \mathbf{g}_i\mathrm{d}u^i\end{aligned} \tag{E.15}$$

where $\mathbf{g}_i$ is the covariant basis of the curvilinear coordinate u^i.

Analogously, the arc length ds between two points of and Q in the curvilinear coordinate system can be calculated by

$$\begin{aligned}(\mathrm{d}s)^2 &= \mathrm{d}\mathbf{r}\cdot\mathrm{d}\mathbf{r}\\ &= (\mathbf{g}_i\mathrm{d}u^i)\cdot(\mathbf{g}_j\mathrm{d}u^j)\\ &= (\mathbf{g}_i\cdot\mathbf{g}_j)\,\mathrm{d}u^i\cdot\mathrm{d}u^j\\ &= g_{ij}\,dx^i\cdot\mathrm{d}x^j \quad \text{for } i,j=1,2,3\end{aligned} \tag{E.16}$$

Therefore,

$$\begin{aligned}\mathrm{d}s &= \sqrt{\left|g_{ij}\,\mathrm{d}x^i\cdot\mathrm{d}x^j\right|}\\ \Rightarrow s &= \int_{t_1}^{t_2}\sqrt{\left|g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\cdot\frac{\mathrm{d}x^j}{\mathrm{d}t}\right|}\mathrm{d}t\end{aligned} \tag{E.17}$$

where

t is the parameter in the curve C with the coordinate $u^i(t)$;

g_{ij} is the defined as the metric coefficient of two non-orthonormal bases.

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \mathbf{g}_j \cdot \mathbf{g}_i = g_{ji} \neq \delta_i^j \tag{E.18}$$

It is obvious that the symmetric metric coefficients g_{ij} vanish for any $i \neq j$ in the orthogonal bases because $\mathbf{g}_i$ is perpendicular to $\mathbf{g}_j$; therefore, the metric tensor can be rewritten as

$$g_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ g_{ii} & \text{if } i = j \end{cases} \tag{E.19}$$

In the orthonormal bases, the metric coefficients g_{ij} in Eq. (E.19) become

$$g_{ij} \equiv \delta_i^j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{E.20}$$

where δ_i^j is called the Kronecker delta.

E.1.5 Bases of the Coordinates

The vector $\mathbf{r}$ can be rewritten in Cartesian coordinates of Euclidean space $\mathbf{E}^3$.

$$\mathbf{r} = x^i \mathbf{e}_i \tag{E.21}$$

The differential d$\mathbf{r}$ results from Eq. (E.21) in

$$\begin{aligned} \mathrm{d}\mathbf{r} &= \mathbf{e}_i \mathrm{d}x^i \\ &= \frac{\partial \mathbf{r}}{\partial x^i} \mathrm{d}x^i \end{aligned} \tag{E.22}$$

Thus, the orthonormal bases $\mathbf{e}_i$ of the coordinate x^i can be defined as

$$\mathbf{e}_{\mathrm{i}} = \frac{\partial \mathbf{r}}{\partial x^i} \quad \text{for } i = 1, 2, 3 \tag{E.23}$$

Analogously, the basis of the curvilinear coordinate u^i can be calculated in the curvilinear coordinate system of $\mathbf{E}^3$.

$$\mathbf{g}_{\mathrm{i}} = \frac{\partial \mathbf{r}}{\partial u^i} \quad \text{for } i = 1, 2, 3 \tag{E.24}$$

Substituting Eq. (E.24) into Eq. (E.18), we obtain the metric coefficients g_{ij} that are generally symmetric in Euclidean space; that is, $g_{ij} = g_{ji}$.

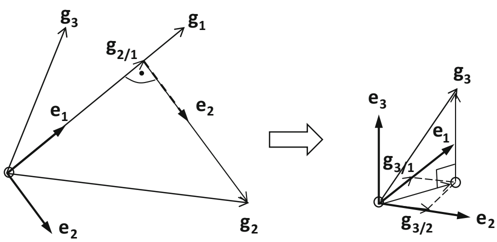

Fig. E.5 Schematic visualization of the Gram–Schmidt procedure

$$\begin{aligned} g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j = g_{ji} \neq \delta_i^j \\ &= \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial \mathbf{r}}{\partial u^j} = \left(\frac{\partial \mathbf{r}}{\partial x^m} \frac{\partial x^m}{\partial u^i} \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial x^n} \frac{\partial x^n}{\partial u^j} \right) \\ &= \frac{\partial x^m}{\partial u^i} \frac{\partial x^n}{\partial u^j} (\mathbf{e}_m . \mathbf{e}_n) = \frac{\partial x^m}{\partial u^i} \frac{\partial x^n}{\partial u^j} \delta_m^n \\ &= \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j} \quad \text{for } k = 1, 2, 3 \end{aligned} \tag{E.25}$$

According to Eqs. (E.4a and E.4b), the contravariant basis $\mathbf{g}^k$ is perpendicular to both covariant bases $\mathbf{g}_i$ and $\mathbf{g}_j$. Additionally, the contravariant basis $\mathbf{g}^k$ is chosen such that the vector length of the contravariant basis equals the inversed vector length of its relating covariant basis; thus, $\mathbf{g}^k \cdot \mathbf{g}_k = 1$. As a result, the scalar products of the covariant and contravariant bases can be written in general curvilinear coordinates $(u^1, \ldots, u^N)$.

$$\begin{cases} \mathbf{g}_i \cdot \mathbf{g}^k = \mathbf{g}^k \cdot \mathbf{g}_i = \delta_i^k & \text{for } i, k = 1, 2, \ldots, N \\ \mathbf{g}_i \cdot \mathbf{g}_k = g_{ik} = g_{ki} \neq \delta_i^k & \text{for } i, k = 1, 2, \ldots, N \end{cases} \tag{E.26}$$

E.1.6 Orthonormalizing a Non-orthonormal Basis

The basis $\{\mathbf{g}_i\}$ is non-orthonormal in the curvilinear coordinates. Using the Gram–Schmidt scheme (Griffiths 2005), an orthonormal basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) can be created from the basis ($\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$). The orthonormalization procedure of the basis ($\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$) will be derived in this section; the orthonormalizing scheme is demonstrated in Fig. E.5.

The Gram–Schmidt scheme for $N = 3$ has three orthonormalization steps:

1. Normalize the first basis vector $\mathbf{g}_1$ by dividing it by its length to get the *normalized* basis $\mathbf{e}_1$.

$$\mathbf{e}_1 = \frac{\mathbf{g}_1}{|\mathbf{g}_1|}$$

2. Project the basis $\mathbf{g}_2$ onto the basis $\mathbf{g}_1$ to get the projection vector $\mathbf{g}_{2/1}$ on the basis $\mathbf{g}_1$. The normalized basis $\mathbf{e}_2$ results from subtracting the projection vector $\mathbf{g}_{2/1}$ from the basis $\mathbf{g}_2$. Then, iteratively, normalize this vector by dividing it by its length to generate the basis $\mathbf{e}_2$.

$$\mathbf{e}_2 = \frac{\mathbf{g}_2 - \mathbf{g}_{2/1}}{\left|\mathbf{g}_2 - \mathbf{g}_{2/1}\right|} = \frac{\mathbf{g}_2 - (\mathbf{g}_2 \cdot \mathbf{e}_1)\mathbf{e}_1}{|\mathbf{g}_2 - (\mathbf{g}_2 \cdot \mathbf{e}_1)\mathbf{e}_1|}$$

3. Subtract projections along the bases of $\mathbf{e}_1$ and $\mathbf{e}_2$ from the basis $\mathbf{g}_3$ and normalize it to obtain the normalized basis $\mathbf{e}_3$.

$$\mathbf{e}_3 = \frac{\mathbf{g}_3 - \mathbf{g}_{3/1} - \mathbf{g}_{3/2}}{\left|\mathbf{g}_3 - \mathbf{g}_{3/1} - \mathbf{g}_{3/2}\right|} = \frac{\mathbf{g}_3 - (\mathbf{g}_3 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{g}_3 \cdot \mathbf{e}_2)\mathbf{e}_2}{|\mathbf{g}_3 - (\mathbf{g}_3 \cdot \mathbf{e}_1)\mathbf{e}_1 - (\mathbf{g}_3 \cdot \mathbf{e}_2)\mathbf{e}_2|}$$

Using the Gram–Schmidt scheme, the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ results from the non-orthonormal bases $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$.

Generally, the orthogonal bases $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$ for the N-dimensional space can be generated from the non-orthonormal bases $\{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_N\}$ according to the Gram–Schmidt scheme as follows:

$$\mathbf{e}_j = \frac{\mathbf{g}_j - \sum_{i=1}^{j-1} (\mathbf{g}_j \cdot \mathbf{e}_i)\mathbf{e}_i}{\left|\mathbf{g}_j - \sum_{i=1}^{j-1} (\mathbf{g}_j \cdot \mathbf{e}_i)\mathbf{e}_i\right|} \quad \text{for } j = 1, 2, \ldots, N$$

E.1.7 Angle Between Two Vectors and Projected Vector Component

The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ can be defined by means of the scalar product (Fig. E.6).

$$\begin{aligned}\cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{\mathbf{g}_i \cdot \mathbf{g}_j a^i b^j}{|\mathbf{a}| \cdot |\mathbf{b}|} \\ &= \frac{g_{ij} a^i b^j}{\sqrt{g_{ij} a^i a^j} \cdot \sqrt{g_{kl} b^k b^l}} = \frac{g_{ij} a^i b^j}{\sqrt{a^i a_i} \cdot \sqrt{b^j b_j}}\end{aligned} \tag{E.27}$$

where

$$\begin{aligned}|\mathbf{a}|^2 &= \mathbf{a} \cdot \mathbf{a} \\ &= g_{ij} a^i a^j = g^{ij} a_i a_j = a^i a_i \quad \text{for } i, j = 1, 2, \ldots, N\end{aligned} \tag{E.28}$$

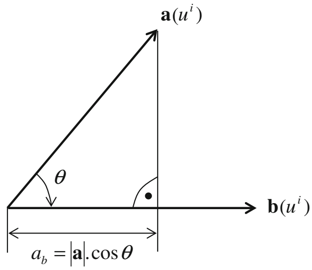

Fig. E.6 Angle between two vectors and projected vector component

in which

a^i, b^j are the contravariant vector components;
a_i, b_j are the covariant vector components;
g_{ij}, g^{ij} are the covariant and contravariant metric coefficients of the bases.

The projected component of the vector **a** on vector **b** results from its vector length and Eq. (E.27).

$$\begin{aligned} a_\mathrm{b} &= |\mathbf{a}| \cdot \cos\theta \\ &= |\mathbf{a}| \cdot \frac{g_{ij}a^i b^j}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{g_{ij}a^i b^j}{|\mathbf{b}|} \\ &= \frac{g_{ij}a^i b^j}{\sqrt{g_{kl}b^k b^l}} \quad \text{for } i,j,k,l = 1,2,\ldots,N \end{aligned} \tag{E.29}$$

Examples
Given two vectors **a** and **b**:

$$\mathbf{a} = 1 \cdot \mathbf{e}_1 + \sqrt{3} \cdot \mathbf{e}_2 = a^i \mathbf{g}_i;$$
$$\mathbf{b} = 1 \cdot \mathbf{e}_1 + 0 \cdot \mathbf{e}_2 = b^j \mathbf{g}_j$$

Thus, the relating vector components are

$$\mathbf{g}_1 = \mathbf{e}_1; \quad \mathbf{g}_2 = \mathbf{e}_2$$
$$a^1 = 1; \quad a^2 = \sqrt{3}$$
$$b^1 = 1; \quad b^2 = 0$$

The covariant metric coefficients g_{ij} in the orthonormal basis ($\mathbf{e}_1$, $\mathbf{e}_2$) can be calculated according to Eq. (E.18).

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The angle θ between two vectors results from Eq. (E.27).

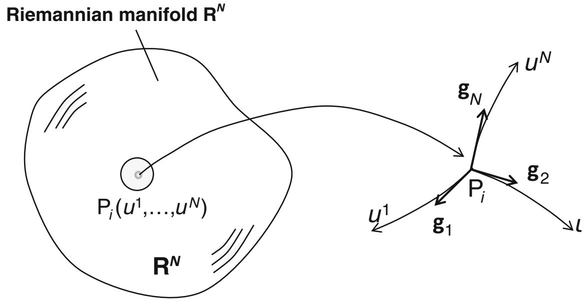

Fig. E.7 *N* tuples of coordinates in Riemannian manifold

$$\cos\theta = \frac{g_{ij}a^i b^j}{\sqrt{g_{ij}a^i a^j}\cdot\sqrt{g_{kl}b^k b^l}} \quad \text{for } i,j,k,l = 1,2$$
$$= \frac{g_{11}a^1b^1 + g_{12}a^1b^2 + g_{21}a^2b^1 + g_{22}a^2b^2}{\sqrt{g_{ij}a^i a^j}\cdot\sqrt{g_{kl}b^k b^l}}$$
$$= \frac{(1\cdot 1\cdot 1) + (0\cdot 1\cdot 0) + (0\cdot\sqrt{3}\cdot 1) + (1\cdot\sqrt{3}\cdot 0)}{\sqrt{1+0+0+3}\cdot\sqrt{1+0+0+0}} = \frac{1}{2}$$

Therefore,

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

The projected vector component can be calculated according to Eq. (E.29).

$$a_b = \frac{g_{ij}a^i b^j}{\sqrt{g_{kl}b^k b^l}} \quad \text{for } i,j,k,l = 1,2$$
$$= \frac{(1\cdot 1\cdot 1) + (0\cdot 1\cdot 0) + (0\cdot\sqrt{3}\cdot 1) + (1\cdot\sqrt{3}\cdot 0)}{\sqrt{1+0+0+0}} = 1$$

E.2 General *N*-dimensional Riemannian Manifold

The concept of the Riemannian geometry is a very important fundamental brick in the modern physics of relativity and quantum field theories, theoretical elementary particles physics, and string theory. In contrast to the homogenous Euclidean manifold, the non-homogenous Riemannian manifold only contains a tuple of fiber bundles of N arbitrary curvilinear coordinates of u^1, …, u^N. Each of the fiber bundle is related to a point and belongs to the N-dimensional differentiable Riemannian manifold. In the case of the infinitesimally small fiber lengths in all dimensions, the fiber bundle now becomes a single point. Therefore, the tuple of fiber bundles becomes a tuple of points in the manifold. In fact, Riemannian manifold only contains a point tuple (Riemann 2013).

In turn, each point of the point tuple can move along a fiber bundle in N arbitrary directions (dimensions) in the N-dimensional Riemannian manifold. Generally, a hypersurface of the fiber bundle of curvilinear coordinates $\{u^i\}$ for $i = 1, 2, \ldots, N$ at a certain point can be defined as a differentiable $(N - 1)$-dimensional subspace with a codimension of one. This definition can be understood that the $(N - 1)$-dimensional subbundle of fibers moves along the one-dimensional remaining fiber.

E.2.1 Point Tuple in Riemannian Manifold

We now consider an N-dimensional differentiable Riemannian manifold $\mathbf{R}^N$ that contains a tuple of points. In general, each point in the manifold locally has N curvilinear coordinates of $u^1, \ldots, u^N$ embedded at this point. Therefore, the considered point P_i can be expressed in the curvilinear coordinates as $P_i(u^1, \ldots, u^N)$. The notation of Riemannian manifold allows the local embedding of an N-dimensional affine tangential manifold (called affine tangential vector space) into the point P_i, as displayed in Fig. E.7. The arc length between any two points of N tuples of coordinates in the manifold does not physically change in any chosen basis. However, its components are changed in the coordinate bases that vary in the manifold. Therefore, these components must be taken into account in the transformation between different curvilinear coordinate systems in Riemannian manifold. To do that, each point in Riemannian manifold can be embedded with the individual metric coefficients g_{ij} for the relating point. Note that the metric coefficients g_{ij} of the coordinates $(u^1, \ldots, u^N)$ at any point are symmetric, and they totally have N^2 components in an N-dimensional manifold. That means one can embed an affine tangential manifold $\mathbf{E}^N$ at any point in Riemannian manifold $\mathbf{R}^N$ in which the metric coefficients g_{ij} could be only applied to this point and change from one point to another point. However, the dot product (inner product) is not valid any longer in the affine tangential manifold (Klingbeil 1966; Riemann 2013).

E.2.2 Flat and Curved Surfaces

By abuse of notation and by completely abstaining from mathematical rigorousness, we introduce the notation of flat and curved surfaces. A surface in Euclidean space is called *flat* if the sum of angles in any triangle ABC is equal to 180° or, alternatively, if the arc length between any two points fulfills the condition in Eq. (E.13). Therefore, the flat surface is a plane in Euclidean space. On the contrary, an arbitrary surface in a Riemannian manifold is called *curved* if the angular sum in an arbitrary triangle ABC is not equal to 180°, as displayed in Fig. E.8.

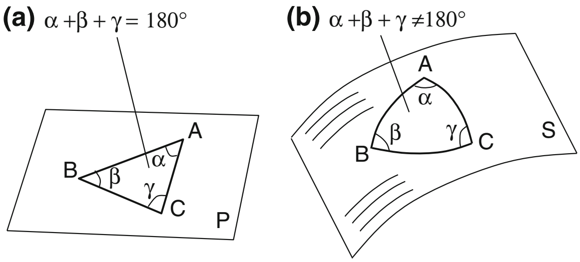

Fig. E.8 Flat and curved surfaces

Conditions for the flat and curved surfaces (Oeijord 2005):

$$\begin{cases} \alpha + \beta + \gamma = 180^\circ & \text{for a flat surface} \\ \alpha + \beta + \gamma \neq 180^\circ & \text{for a curved surface} \end{cases} \tag{E.30}$$

Furthermore, the surface curvature in Riemannian manifold can be used to determine the surface characteristics. Additionally, the line curvature is also applied to studying the curve and surface characteristics.

E.2.3 Arc Length Between Two Points in Riemannian Manifold

We now consider a differentiable Riemannian manifold and calculate the arc length between two points $P(u^1, \ldots, u^N)$ and $Q(u^1, \ldots, u^N)$. The arc length is an important notation in Riemannian manifold theory. The coordinates $(u^1, \ldots, u^N)$ can be considered as a function of the parameter t that varies from $P(t_1)$ to $Q(t_2)$.

The *arc length* ds between the points P and Q thus results from

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \tag{E.31}$$

where the derivative of the vector $\mathbf{r}(u^1, \ldots, u^N)$ can be calculated as

$$\begin{aligned} \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= \frac{\mathrm{d}(\mathbf{g}_i u^i)}{\mathrm{d}t} \\ &\equiv \mathbf{g}_i \dot{u}^i(t) \quad \text{for } i = 1, 2, \ldots, N \end{aligned} \tag{E.32}$$

Substituting Eq. (E.32) into Eq. (E.31), one obtains the arc length

$$\begin{aligned} \mathrm{d}s &= \sqrt{\varepsilon (\mathbf{g}_i \dot{u}^i) \cdot (\mathbf{g}_j \dot{u}^j)}\mathrm{d}t \\ &= \sqrt{\varepsilon\, g_{ij} \dot{u}^i(t) \dot{u}^j(t)}\mathrm{d}t \quad \text{for } i, j = 1, 2, \ldots, N \end{aligned} \tag{E.33}$$

where ε (= ±1) is the functional indicator that ensures the square root always exists.

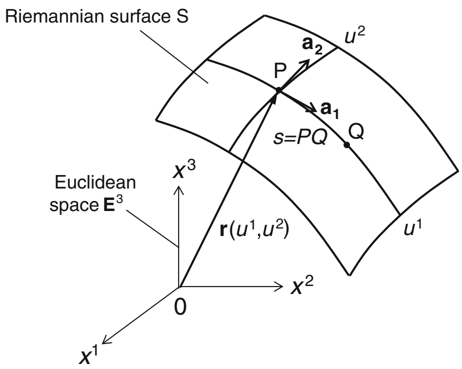

Fig. E.9 Arc length between two points in a Riemannian surface

Therefore, the arc length of PQ is given by integrating Eq. (E.33) from the parameter t_1 to the parameter t_2.

$$s = \int_{t_1}^{t_2} \sqrt{\varepsilon \, g_{ij} \dot{u}^i(t) \dot{u}^j(t)} \, \mathrm{d}t \quad \text{for } i,j = 1,2,\ldots,N \tag{E.34}$$

where the covariant metric coefficients g_{ij} are defined by

$$\begin{aligned} g_{ij} &= \mathbf{g}_i \cdot \mathbf{g}_j \neq \delta_i^j \\ &= \frac{\partial x^k}{\partial u^i} \cdot \frac{\partial x^k}{\partial u^j} \quad \text{for } k = 1,2,\ldots,N \end{aligned} \tag{E.35}$$

We now assume that the points $P(u^1,u^2)$ and $Q(u^1,u^2)$ lie on the Riemannian surface S, which is embedded in Euclidean space $\mathbf{E}^3$. Each point on the surface only depends on two parameterized curvilinear coordinates of u^1 and u^2 that are called the Gaussian surface parameters, as shown in Fig. E.9.

The differential d$\mathbf{r}$ of the vector $\mathbf{r}$ can be rewritten in the coordinates (u^1, u^2):

$$\begin{aligned} \mathrm{d}\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u^i} \mathrm{d}u^i \\ &\equiv \mathbf{r}_{,i} \mathrm{d}u^i \\ &\equiv \mathbf{a}_i \mathrm{d}u^i \quad \text{for } i = 1,2 \end{aligned} \tag{E.36}$$

where $\mathbf{a}_i$ is the tangential vector of the coordinate u^i on the Riemannian surface.

Therefore, the arc length ds on the differentiable Riemannian parameterized surface can be computed as

$$\begin{aligned} (\mathrm{d}s)^2 &= \mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} \\ &= \mathbf{a}_i \cdot \mathbf{a}_j \mathrm{d}u^i \mathrm{d}u^j \\ &\equiv a_{ij} \mathrm{d}u^i \mathrm{d}u^j \quad \text{for } i,j = 1,2 \end{aligned} \tag{E.37}$$

whereas a_{ij} are the surface metric coefficients only at the point P in the coordinates (u^1, u^2) on the Riemannian curved surface S. The formulation of $(\mathrm{d}s)^2$ in Eq. (E.37) is called the first fundamental form for the intrinsic geometry of Riemannian manifold (Springer 2012; Lang 1999; Lee 2000; Fecko 2011).

The surface metric coefficients of the covariant and contravariant components have the similar characteristics such as the metric coefficients:

$$\begin{aligned} a_{ij} &= a_{ji} = \mathbf{a}_i \cdot \mathbf{a}_j \neq \delta_i^j \\ &= \frac{\partial x^k}{\partial u^i} \cdot \frac{\partial x^k}{\partial u^j} \quad \text{for } k = 1, 2, \ldots, N; \end{aligned} \tag{E.38a}$$

$$a_i^j = \mathbf{a}_i \cdot \mathbf{a}^j = \delta_i^j \tag{E.38b}$$

Instead of the metric coefficients g_{ij} in the curvilinear Euclidean space, the surface metric coefficients a_{ij} are used in the general curvilinear Riemannian manifold.

E.2.4 Tangent and Normal Vectors on the Riemannian Surface

We consider a point $P(u^1, u^2)$ on a differentiable Riemannian surface that is parameterized by u^1 and u^2. Furthermore, the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are the covariant bases of the curvilinear coordinates (u^1, u^2), respectively. In general, a hypersurface in an N-dimensional manifold with coordinates $\{u^i\}$ for $i = 1, 2, \ldots, N$ can be defined as a differentiable $(N - 1)$-dimensional subspace with a codimension of 1.

The basis $\mathbf{a}_i$ of the coordinate u^i can be rewritten as

$$\begin{aligned} \mathbf{a}_i &= \frac{\partial \mathbf{r}}{\partial u^i} \\ &\equiv \mathbf{r}_{,i} \quad \text{for } i = 1, 2 \end{aligned} \tag{E.39}$$

The covariant basis $\mathbf{a}_i$ is tangent to the coordinate u^i at the point P. Both bases $\mathbf{a}_1$ and $\mathbf{a}_2$ generate the tangential surface T tangent to the Riemannian surface S at the point P, which is defined by the curvilinear coordinates of u^1 and u^2, as shown in Fig. E.10.

The angle of two intersecting Gaussian parameterized curves u^i and u^j results from the dot product of the bases at the point $P(u^i, u^j)$.

$$\begin{aligned} &\mathbf{a}_i \cdot \mathbf{a}_j = |\mathbf{a}_i| \cdot |\mathbf{a}_j| \cos(\mathbf{a}_i, \mathbf{a}_j) \Rightarrow \\ &\cos\theta_{ij} \equiv \cos(\mathbf{a}_i, \mathbf{a}_j) = \frac{\mathbf{a}_i \cdot \mathbf{a}_j}{|\mathbf{a}_i| \cdot |\mathbf{a}_j|} \\ &\qquad = \frac{a_{ij}}{\sqrt{a_{(ii)}} \cdot \sqrt{a_{(jj)}}} \leq 1 \quad \text{for } \theta_{ij} \in \left[0, \frac{\pi}{2}\right] \end{aligned} \tag{E.40}$$

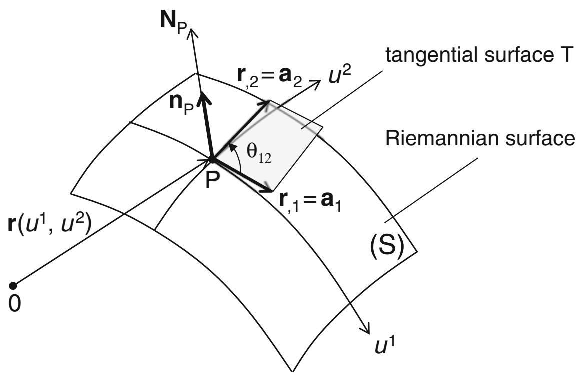

Fig. E.10 Tangent vectors to the curvilinear coordinates (u^1, u^2)

Note that

$$|\mathbf{a}_i| = \sqrt{\mathbf{a}_i \cdot \mathbf{a}_i} = \sqrt{a_{(ii)}}, \text{ no summation over (ii)}$$

where a_{ii} and a_{jj} are the vector lengths of $\mathbf{a}_i$ and $\mathbf{a}_j$; a_{ij}, the surface metric coefficients.

The surface metric coefficients can be defined by

$$\begin{aligned} a_{ij} &= \mathbf{a}_i \cdot \mathbf{a}_j \neq \delta_i^j \\ &= \frac{\partial x^k}{\partial u^i} \cdot \frac{\partial x^k}{\partial u^j} \quad \text{for } k = 1, 2\ldots, N \end{aligned} \tag{E.41}$$

E.2.5 Angle Between Two Curvilinear Coordinates

We now give a concrete example of the computation of the angle between two curvilinear coordinates. Given two arbitrary basis vectors at the point $P(u^1, u^2)$, we can write them with the covariant basis $\{\mathbf{e}_i\}$:

$$\mathbf{a}_1 = 1 \cdot \mathbf{e}_1 + 0 \cdot \mathbf{e}_2;$$
$$\mathbf{a}_2 = 0 \cdot \mathbf{e}_1 + 1 \cdot \mathbf{e}_2.$$

The covariant metric coefficients a_{ij} can be calculated:

$$(\mathbf{a}_{\text{ij}}) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The angle between two base vectors results from Eq. (E.40):

$$\cos\,\theta_{ij} = \frac{a_{ij}}{\sqrt{a_{(ii)}} \cdot \sqrt{a_{(jj)}}}$$
$$\Rightarrow \cos\,\theta_{12} = \frac{a_{12}}{\sqrt{a_{11}} \cdot \sqrt{a_{22}}} = \frac{0}{\sqrt{1} \cdot \sqrt{1}} = 0$$

Thus,

$$\theta_{12} = \cos^{-1}\left(\frac{a_{12}}{\sqrt{a_{11}} \cdot \sqrt{a_{22}}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

In this case, the curvilinear coordinates of u^1 and u^2 are orthogonal at the point P on the Riemannian surface S, as shown in Fig. E.10.

The tangent vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ generate the tangential surface T tangent to the Riemannian surface S at the point P. The normal vector $\mathbf{N}_P$ to the tangential surface T at the point P is given by

$$\begin{aligned} \mathbf{N}_P &= \frac{\partial \mathbf{r}}{\partial u^i} \times \frac{\partial \mathbf{r}}{\partial u^j} = \mathbf{r}_{,i} \times \mathbf{r}_{,j} \\ &\equiv \mathbf{a}_i \times \mathbf{a}_j \quad \text{for } i,j = 1,2 \\ &= \alpha\,\mathbf{a}^k \end{aligned} \tag{E.42}$$

where

α is the scalar factor;

$\mathbf{a}^k$ is the contravariant basis of the curvilinear coordinate of u^k.

Multiplying Eq. (E.42) by the covariant basis $\mathbf{a}_k$, the scalar factor α results in

$$\begin{aligned} &\alpha(\mathbf{a}^k \cdot \mathbf{a}_k) = \alpha\,\delta_k^k = \alpha = (\mathbf{a}_i \times \mathbf{a}_j) \cdot \mathbf{a}_k \\ &\Rightarrow \alpha = (\mathbf{a}_i \times \mathbf{a}_j) \cdot \mathbf{a}_k \equiv [\mathbf{a}_i, \mathbf{a}_j, \mathbf{a}_k] \end{aligned} \tag{E.43}$$

The scalar factor α equals the scalar triple product that is given in Nayak (2012):

$$\begin{aligned} \alpha &\equiv [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] = (\mathbf{a}_i \times \mathbf{a}_j) \cdot \mathbf{a}_k = (\mathbf{a}_k \times \mathbf{a}_i) \cdot \mathbf{a}_j = (\mathbf{a}_j \times \mathbf{a}_k) \cdot \mathbf{a}_i \\ &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{vmatrix}^{\frac{1}{2}} \\ &= \sqrt{\det(\mathbf{a}_{ij})} \equiv J \end{aligned} \tag{E.44}$$

where Jacobian J is the determinant of the covariant basis tensor.

The unit normal vector $\mathbf{n}_P$ in Eq. (E.42) becomes using the Lagrange identity.

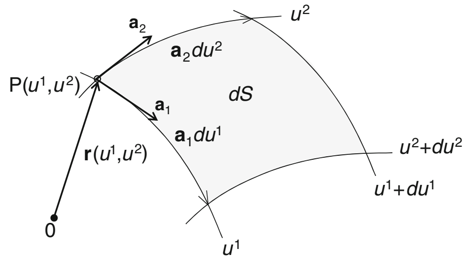

Fig. E11 Surface area in the curvilinear coordinates

$$\mathbf{n}_P = \frac{\mathbf{a}_i \times \mathbf{a}_j}{|\mathbf{a}_i \times \mathbf{a}_j|} = \frac{\mathbf{a}_i \times \mathbf{a}_j}{\sqrt{a_{(ii)} \cdot a_{(jj)} - (a_{ij})^2}} \tag{E.45}$$

Note that

$$|\mathbf{a}_i|^2 = \mathbf{a}_i \cdot \mathbf{a}_i = a_{(ii)}, \text{ no summation over (ii)}$$

The Lagrange identity results from the cross product of two vectors **a** and **b**.

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}| \cdot |\mathbf{b}| \sin(\mathbf{a}, \mathbf{b}) \Rightarrow \\
|\mathbf{a} \times \mathbf{b}|^2 &= |\mathbf{a}|^2 \cdot |\mathbf{b}|^2 \sin^2(\mathbf{a}, \mathbf{b}) \\
&= |\mathbf{a}|^2 \cdot |\mathbf{b}|^2 \cdot \left(1 - \cos^2(\mathbf{a}, \mathbf{b})\right) \\
&= (|\mathbf{a}| \cdot |\mathbf{b}|)^2 - (|\mathbf{a}|.|\mathbf{b}| \cdot \cos(\mathbf{a}, \mathbf{b}))^2 \\
&= (|\mathbf{a}| \cdot |\mathbf{b}|)^2 - (\mathbf{a} \cdot \mathbf{b})^2
\end{aligned}$$

Thus,

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{|\mathbf{a}|^2 \cdot |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \tag{E.46}$$

Equation (E.46) is called the *Lagrange identity*.

E.2.6 Surface Area in Curvilinear Coordinates

The surface area S in the differentiable Riemannian curvilinear surface, as displayed in Fig. E.11, can be calculated using the Lagrange identity (Nayak 2012).

$$
\begin{aligned}
S &= \iint \left| \frac{\partial \mathbf{r}}{\partial u^i} \times \frac{\partial \mathbf{r}}{\partial u^j} \right| \mathrm{d}u^i \mathrm{d}u^j \\
&= \iint \left| \mathbf{a}_i \times \mathbf{a}_j \right| \mathrm{d}u^i \mathrm{d}u^j \\
&= \iint \sqrt{|\mathbf{a}_i|^2 . \, |\mathbf{a}_j|^2 - (\mathbf{a}_i . \mathbf{a}_j)^2} \, \mathrm{d}u^i \mathrm{d}u^j \\
&= \iint \sqrt{a_{(ii)} . \, a_{(jj)} - (a_{ij})^2} \, \mathrm{d}u^i \mathrm{d}u^j
\end{aligned}
\tag{E.47}
$$

Therefore,

$$
\begin{aligned}
S &= \iint |\mathbf{a}_1 \times \mathbf{a}_2| \, \mathrm{d}u^1 \mathrm{d}u^2 \quad \text{for } i = 1; j = 2 \\
&= \iint \sqrt{a_{11} \cdot a_{22} - (a_{12})^2} \mathrm{d}u^1 \mathrm{d}u^2
\end{aligned}
\tag{E.48}
$$

In Eq. (E.47), the vector length squared can be calculated as

$$
|\mathbf{a}_i|^2 = \mathbf{a}_i \cdot \mathbf{a}_i = a_{(ii)} \text{ no summation over (ii)}
$$

E.3 Kronecker Delta

The Kronecker delta is very useful in tensor analysis and is defined as

$$
\delta_i^j \equiv \frac{\partial u^j}{\partial u^i} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}
\tag{E.49}
$$

where u^i and u^j are in the same coordinate system and independent of each other.

Some properties of the Kronecker delta (Kronecker tensor) are considered (Nayak 2012; Oeijord 2005). We summarize a few properties of the Kronecker delta:

Property 1
Chain rule of differentiation of the Kronecker delta using the contraction rule (cf. Appendix A)

$$
\delta_i^j = \frac{\partial u^j}{\partial u^i} = \frac{\partial u^j}{\partial u^k} \frac{\partial u^k}{\partial u^i} = \delta_k^j \delta_i^k
\tag{E.50}
$$

Property 2
Kronecker delta in Einstein summation convention

$$\begin{aligned} \delta^i_j a^{jk} &= \delta^i_1 a^{1k} + \cdots + \delta^i_i a^{ik} + \cdots + \delta^i_N a^{Nk} \\ &= 0 + \cdots + 1.a^{ik} + \cdots + 0 \\ &= a^{ik} \end{aligned} \tag{E.51}$$

Property 3
Product of Kronecker deltas

$$\begin{aligned} \delta^i_j \delta^j_k &= \delta^i_1 \delta^1_k + \cdots + \delta^i_i \delta^i_k + \cdots + \delta^i_N \delta^N_k \\ &= 0 + \cdots + 1.\delta^i_k + \cdots + 0 \\ &= \delta^i_k \end{aligned} \tag{E.52}$$

Note that

$$\delta^{(i)}_{(i)} \equiv \delta^1_1 = \delta^2_2 = \cdots = \delta^N_N = 1 \text{ (no summation over the free index } i\text{);}$$

However,

$$\delta^i_i \equiv \delta^1_1 + \delta^2_2 + \cdots + \delta^N_N = N \text{ (summation over the dummy index } i\text{).}$$

E.4 Levi-Civita Permutation Symbols

Levi-Civita permutation symbols in a three-dimensional space are third-order pseudo-tensors. They are a useful tool to simplify the mathematical expressions and computations (Klingbeil 1966; Nayak 2012; Kay 2011).

The Levi-Civita permutation symbols can simply be defined as

$$\begin{aligned} \varepsilon_{ijk} &= \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation;} \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation;} \\ 0 & \text{if } i = j, \text{ or } i = k; \text{ or } j = k \end{cases} \\ \Leftrightarrow \varepsilon_{ijk} &= \frac{1}{2}(i-j)\cdot(j-k)\cdot(k-i) \quad \text{for } i,j,k = 1,2,3 \end{aligned} \tag{E.53}$$

Here, we abstain from giving an exact definition of even and odd permutations because this would go beyond the scope of this book. The reader is referred to the literature (Lee 2000; Fecko 2011).

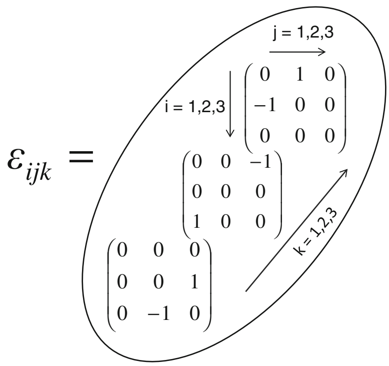

Fig. E.12 27 Levi-Civita permutation symbols

According to Eq. (E.53), the Levi-Civita permutation symbols can be expressed as

$$\varepsilon_{ijk} = \begin{cases} \varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} \text{ (even permutation)}; \\ -\varepsilon_{ikj} = -\varepsilon_{kji} = -\varepsilon_{jik} \text{ (odd permutation)} \end{cases} \tag{E.54}$$

The 27 Levi-Civita permutation symbols for a three-dimensional coordinate system are graphically displayed in Fig. E.12.

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Definitions of Mathematical Symbols in this Book

- First partial derivative of a second-order tensor with respect to u^k

$$T^{ij}_{,k} \equiv \frac{\partial T^{ij}}{\partial u^k} \tag{1}$$

Do not confuse Eq. (1) with the symbol used in some books:

$$T^{ij}_{,k} \equiv \frac{\partial T^{ij}}{\partial u^k} + \Gamma^i_{km} T^{mj} + \Gamma^j_{km} T^{im}$$

This symbol is equivalent to Eq. (2) used in this book.

- First covariant derivative of a second-order tensor with respect to u^k

$$T^{ij}|_k \equiv T^{ij}_{,k} + \Gamma^i_{km} T^{mj} + \Gamma^j_{km} T^{im} \tag{2}$$

- Second partial derivative of a first-order tensor with respect to u^j and u^k

$$T_{i,jk} \equiv \frac{\partial^2 T_i}{\partial u^j \partial u^k} \tag{3}$$

Do not confuse Eq. (3) with the symbol used in some books:

$$T_{i,jk} \equiv \frac{\partial^2 T_i}{\partial u^j \partial u^k} - \Gamma^m_{ik,j} T_m - \Gamma^m_{ik} \frac{\partial T_m}{\partial u^j} - \Gamma^m_{ij} \frac{\partial T_m}{\partial u^k} + \Gamma^m_{ij} \Gamma^n_{mk} T_n - \Gamma^m_{jk} \frac{\partial T_i}{\partial u^m} + \Gamma^m_{jk} \Gamma^n_{im} T_n$$

This symbol is equivalent to Eq. (4) used in this book.

- Second covariant derivative of a first-order tensor with respect to u^j and u^k

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$$T_i\big|_{kj} \equiv T_{i,jk} - \Gamma^m_{ik,j}T_m - \Gamma^m_{ik}T_{m,j} - \Gamma^m_{ij}T_{m,k} + \Gamma^m_{ij}\Gamma^n_{mk}T_n - \Gamma^m_{jk}T_{i,m} + \Gamma^m_{jk}\Gamma^n_{im}T_n \quad (4)$$

- Christoffel symbols of first kind

  Γ_{ijk}instead of $[i,j,k]$ used in some books.

- Christoffel symbols of second kind

  Γ^k_{ij}instead of $\left\{ {k \atop i \quad j} \right\}$ used in some books.

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